

Mac Williams identities and polarized Riemann-Roch conditions ^{*}

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Abstract

The present note establishes the equivalence of Mac Williams identities for an additive code C and its dual C^\perp to Polarized Riemann-Roch Conditions on their ζ -functions. In such a way, the duality of additive codes appears to be a polarized form of the Serre duality on a smooth irreducible projective curve.

1 Introduction

Let $(G, +)$ be a finite abelian group and (\widehat{G}, \cdot) be the group of the multiplicative characters $\pi : (G, +) \rightarrow (\mathbb{C}^*, \cdot)$ of G . The subgroups $(C, +)$ of $(G^n, +)$ are called additive codes. Any linear code $C \subset \mathbb{F}_q^n$ over a finite field \mathbb{F}_q is an additive code in the n -th Cartesian power of the finite abelian group $(\mathbb{F}_q, +) \simeq (\mathbb{Z}_p^m, +)$ with $p = \text{char}\mathbb{F}_q$, $q = p^m$. The dual code

$$C^\perp := \{\pi = (\pi_1, \dots, \pi_n) \in \widehat{G}^n \mid \pi(a) = 1, \forall a \in C\}$$

is a subgroup (C^\perp, \cdot) of $(\widehat{G}^n, \cdot) = (\widehat{G^n}, \cdot)$ and can be viewed as an additive code over \widehat{G} . If $\varepsilon : G \rightarrow \mathbb{C}^*$ is the trivial character with $\varepsilon(g) = 1$ for $\forall g \in G$ then the Hamming weight on G and \widehat{G} are defined as

$$\text{wt} : G \longrightarrow \{0, 1\}, \quad \text{wt}(g) := \begin{cases} 0 & \text{if } g = 0_G, \\ 1 & \text{if } g \neq 0_G, \end{cases} \quad (1)$$

respectively,

$$\text{wt} : \widehat{G} \longrightarrow \{0, 1\}, \quad \text{wt}(\pi) := \begin{cases} 0 & \text{if } \pi = \varepsilon, \\ 1 & \text{if } \pi \neq \varepsilon. \end{cases} \quad (2)$$

For an arbitrary $n \in \mathbb{N}$, these extend to

$$\text{wt} : G^n \longrightarrow \{0, 1, \dots, n\}, \quad \text{wt}(a_1, \dots, a_n) := \sum_{i=1}^n \text{wt}(a_i), \quad \text{respectively} \quad (3)$$

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$$\text{wt} : \widehat{G}^n \longrightarrow \{0, 1, \dots, n\}, \quad \text{wt}(\pi_1, \dots, \pi_n) := \sum_{i=1}^n \text{wt}(\pi_i). \quad (4)$$

That enables to define the homogeneous weight enumerator

$$\mathcal{W}_C(x, y) := \sum_{c \in C} x^{n-\text{wt}(c)} y^{\text{wt}(c)}$$

of an additive code $(C, +) \leq (G^n, +)$. Similarly, the dual code

$$C^\perp := \{\pi = (\pi_1, \dots, \pi_n) \in \widehat{G}^n \mid \pi(a) = 1, \forall a \in C\} \simeq (\widehat{G^n/C}, \cdot)$$

has homogeneous weight enumerator

$$\mathcal{W}_{C^\perp} := \sum_{\pi \in C^\perp} x^{n-\text{wt}(\pi)} y^{\text{wt}(\pi)}.$$

According to [2] (see also [20] or [6]), Fourier inversion formula for the function

$$F : (G^n/C, +) \longrightarrow \mathbb{C}[x, y]^{(n)},$$

associating to a coset $a + C \in (G^n/C, +)$ its homogeneous weight enumerator $F(a + C)$ provides Mac Williams identities

$$\mathcal{W}_{C^\perp}(x, y) = \frac{1}{|C|} \mathcal{W}_C(x + (|G| - 1)y, x - y)$$

for $\mathcal{W}_C(x, y)$, $\mathcal{W}_{C^\perp}(x, y)$.

Mac Williams initiates the study of the duality of linear codes by their weight distributions with respect to the Hamming weight in [11]. Delsarte generalizes Mac Williams results in [3] by the means of association schemes. Zinoviev and Ericson's [21] describes Mac Williams duality for additive codes $(C, +) \leq (G^n, +)$ and their duals $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ with respect to isomorphic partitions of G^n and \widehat{G}^n . In [6] Gluering-Luerssen proves Mac Williams identities for $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ and $(C, +) \leq (G^n, +)$ with respect to an arbitrary partition with M blocks on \widehat{G}^n and its Fourier transform, which is a partition on G^n . In the case of $M = 2$, her set up reduces to the Hamming weights on G, \widehat{G} and G^n, \widehat{G}^n . Mac Williams identities over finite Frobenius rings are studies by Greferath-Schmidt's [7], Honold-Landjev's [8], Wood's [20], etc.

Let $C \subset \mathbb{F}_q^n$ be an \mathbb{F}_q -linear $[n, k, d]$ -code with dual

$$C^\perp := \{a = (a_1, \dots, a_n) \in \mathbb{F}_q^n \mid \langle a, c \rangle = \sum_{i=1}^n a_i c_i = 0, \quad \forall c \in C\}$$

of minimum distance d^\perp . The deviation $g := n + 1 - d - k \in \mathbb{Z}^{\geq 0}$ of the parameters of C from the equality in the Singleton bound is called the genus of C . In [4], [5] Duursma introduces the ζ -polynomials $P_C(t), P_{C^\perp}(t) \in \mathbb{Q}[t]$ of degree $\deg P_C(t) = \deg P_{C^\perp}(t) = g + g^\perp = n + 2 - d - d^\perp$ and shows that Mac Williams identities for C, C^\perp are equivalent to the functional equation

$$P_{C^\perp}(t) = P_C\left(\frac{1}{qt}\right) q^g t^{g+g^\perp} \quad (5)$$

for their ζ -polynomials. Note that (5) is a polarized form of the functional equation of the Hasse-Weil polynomial of a smooth irreducible projective curve of genus g , defined over \mathbb{F}_q . The article [17] of Pellikaan, Shen and van Wee sheds a light on this phenomenon. More precisely, [17] shows that for an arbitrary \mathbb{F}_q -linear code $C \subset \mathbb{F}_q^n$ there is a smooth irreducible projective curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$, defined over \mathbb{F}_q , distinct rational points $P_1, \dots, P_n \in X(\mathbb{F}_q) := X \cap \mathbb{P}^N(\mathbb{F}_q)$ and a divisor G of $\mathbb{F}_q(X)$, whose support is disjoint from the support of $D = P_1 + \dots + P_n$, such that $X = \mathcal{E}_D H^0(X, \mathcal{O}_X([G]))$ coincides with the image of the evaluation map

$$\mathcal{E}_D : H^0(X, \mathcal{O}_X([G])) = \mathcal{L}_X(G) \longrightarrow \mathbb{F}_q^n, \quad \mathcal{E}_D(f) = (f(P_1), \dots, f(P_n)) \quad \text{for } \forall f \in \mathcal{L}_X(G).$$

The kernel of \mathcal{E}_D coincides with $\mathcal{L}_X(G - D)$ (cf. Proposition 17 (i) from section 5) and C is isomorphic to the quotient space $\mathcal{L}_X(G)/\mathcal{L}_X(G - D)$ as a linear space over \mathbb{F}_q . The dual code C^\perp is isomorphic to the quotient space $\mathcal{L}_X(K_X - G + D)/\mathcal{L}_X(K_X - G)$, where K_X stands for a canonical divisor of X . Under the Serre duality on X , the first cohomology group $H^1(X, \mathcal{O}_X([E]))$ is isomorphic to the sections $\mathcal{L}_X(K_X - E) = H^0(X, \mathcal{O}_X([K_X - E]))$ of the line bundle $\mathcal{O}_X([E])$, corresponding to the divisor $K_X - E$. If we view the divisor $K_X - E$ as a Serre dual of E then the presentations $C \simeq \mathcal{L}_X(G)/\mathcal{L}_X(G - D)$ and $C^\perp \simeq \mathcal{L}_X(K_X - G + D)/\mathcal{L}_X(K_X - G)$ of mutually dual linear codes are compatible with the Serre duality on X . From now on, let $l(E) : \dim_{\mathbb{F}_q} \mathcal{L}_X(E)$ be the dimension of the space $\mathcal{L}_X(E) = H^0(X, \mathcal{O}_X([E]))$ of the global sections of $\mathcal{O}_X([E])$. The Riemann-Roch Theorem on X is a numerical expression of the difference $l(G) - l(K_X - G)$ by topological invariants of X, G , i.e., by the genus g of X and the degree m of G . Thus, it is reasonable the numerical relation between the weight distributions of C, C^\perp , provided by Mac Williams identities to be compatible with the Serre duality on X and to play the role of the Riemann-Roch Theorem for C, C^\perp .

The relation between the local Weil ζ -function $\zeta_X(t)$ of X and Duursma's ζ -functions $\zeta_{C_i}(t) := \frac{P_{C_i}(t)}{(1-t)(1-qt)}$ of the linear codes $C_i = \mathcal{E}_D \mathcal{L}_X(G_i)$, associated with a complete set of representatives $G_i, 1 \leq i \leq h$ of the linear equivalence classes of the divisors of $\mathbb{F}_q(X)$ of degree $2g \leq m < n$ is noticed by Duursma in [4], [5]. However, the algebraic-geometric representations $C = \mathcal{L}_X(G)$ of arbitrary linear codes $C \subset \mathbb{F}_q^n$, constructed by Pellikaan, Shen and van Wee in [?] tend to have $g > m > n$. As a result, if there exist G_i with $\text{Supp}(G_i) \cap \text{Supp}(D) = \emptyset$ for $\forall 1 \leq i \leq h$ then the ζ 0functions of $C_i = \mathcal{E}_D \mathcal{L}_X(G_i)$ are related with the truncated local Weil ζ -function $\zeta_X^{(m)}(t) = \frac{P_X^{(m)}(t)}{(1-t)(1-qt)}$ of X . (If $P_X(t) \in \mathbb{Z}[t]$ is the Hasse-Weil polynomial of X of degree $2g$ then $P_X^{(m)}(t)$ is the sum of the terms of $P_X(t)$ of degree $\leq m$.) Besides, if a linear equivalence class $G_i + \text{div}_{\mathbb{F}_q}(X)$ of divisors of $\mathbb{F}_q(X)$ of degree m has no representative G_i with $\text{Supp}(G_i) \cap \text{Supp}(D) = \emptyset$ then the evaluation map \mathcal{E}_D at D does not act on $G_i + \text{div}_{\mathbb{F}_q}(X)$ and the available ζ -functions of algebraic-geometric codes do not reflect the information for the effective divisors from $G_i + \text{div}_{\mathbb{F}_q}(X)$. For a detailed discussion of this kind of problems see [14].

The aim of the present note is to understand Mac Williams duality of additive codes in terms of algebraic geometry. It is completely independent and of a different, more formal nature than the recent work [18] of Randriambololona. Note that the Riemann-Roch Theorem 44 for linear codes $C, C^\perp \subset \mathbb{F}_q^n$ from [18] is stronger than our Polarized Riemann-Roch Conditions $\text{PRRC}_q(g, g^\perp)$ on $\zeta_C(t), \zeta_{C^\perp}(t)$, as far as it implies the func-

tional equation on $\zeta_C(t), \zeta_{C^\perp}(t)$, which we show to be equivalent to $\text{PRRC}_q(g, g^\perp)$. Mac Williams identities are used in Delsarte's [3], Byrne-Greferath-Sullivan's [1] and other works for obtaining linear programming bounds on codes. For applications in the engineering one can see ElKhamy-McEliece's [15] or Lu-Kumar-Yang's [10].

The main result of the present article is the equivalence of Mac Williams identities for additive codes $(C, +) \leq (G^n, +)$, $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ to Polarized Riemann-Roch Conditions for their ζ -functions

$$\zeta_C(t) := \frac{P_C(t)}{(1-t)(1-|G|t)}, \quad \zeta_{C^\perp}(t) := \frac{P_{C^\perp}(t)}{(1-t)(1-|G|t)}$$

In such a way, Mac Williams duality of additive codes turns to be a polarized form of Serre duality from algebraic geometry. A crucial step from the proof of the aforementioned equivalence is the study of the additive MDS-codes $(\text{MDS}(n, d), +) \leq (G^n, +)$ of length n and minimum distance d , defined as the ones of genus $g = n+1-d-\log_{|G|}(|\text{MDS}(n, d)|) = 0$. After showing that for any $(n-k)$ -tuple of indices $\beta = \{\beta_1, \dots, \beta_{n-k}\} \subsetneq [n] := \{1, \dots, n\}$ the puncturing (or erasing) $\Pi_\beta : (\text{MDS}(n, d), +) \rightarrow (G^k, +)$ of the components, labeled by β is an isomorphism, we compute explicitly the homogeneous weight enumerator $\mathcal{M}_{n,d}(x, y)$ of $\text{MDS}(n, d)$ and observe that it depends only on n and d .

Here is a synopsis of the article. In section 2 we study the additive codes of genus 0, called the additive MDS-codes. After showing that the dual of an additive MDS-code $\text{MDS}(n, d)$ of length n and minimum distance $d > 1$ is an additive MDS-code $\text{MDS}(n, n+2-d)$ of length n and minimum distance $n+2-d$, we establish that for any unordered d -tuple $\gamma \in \binom{[n]}{d}$ with entries from $[n]$ there are exactly $|G| - 1$ words of $\text{MDS}(n, d)$ with support γ . Then we show that the shortening of the dual $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ of an arbitrary additive code $(C, +) \leq (G^n, +)$ at some component coincides with the puncturing of C at that component. That enables to obtain explicitly the number $\mathcal{M}_{n,d}^{(s)}$ of the words of weight $d \leq s \leq n$ in an additive MDS-code $\text{MDS}(n, d)$. The third section introduces the ζ -polynomial $P_C(t) \in \mathbb{Q}[t]$ and Duursma's reduced polynomial $D_C(t) \in \mathbb{Q}[t]$ of an additive code $(C, +) \leq (G^n, +)$ and expresses Mac Williams identities for C, C^\perp as functional equations on $P_C(t), P_{C^\perp}(t)$ or, respectively, on $D_C(t), D_{C^\perp}(t)$. The fourth section expresses the Riemann-Roch Theorem on a smooth irreducible projective curve X of genus $g \geq 0$, defined over a finite field \mathbb{F}_q as (non-polarized) Riemann-Roch Conditions with base $q \in \mathbb{N}$ and genus g on the local Weil ζ -function $\zeta_X(t)$ of X . That motivates the notion of Polarized Riemann-Roch Conditions $\text{PRRC}_q(g, g^\perp)$ with base $q \in \mathbb{N}$ and genera $g, g^\perp \in \mathbb{Z}^{\geq 0}$ on a pair $\zeta(t), \zeta^\perp(t) \in \mathbb{C}[[t]]$ of formal power series in one variable t . The functional equation for $D_C(t), D_{C^\perp}(t) \in \mathbb{Q}[t]$, expressing Mac Williams identities for the weight distribution of C, C^\perp of genera g, g^\perp is shown to be equivalent to the Polarized Riemann-Roch Conditions $\text{PRRC}_{|G|}(g, g^\perp)$ on the ζ -functions $\zeta_C(t),$

$\zeta_{C^\perp}(t)$. As a consequence, the lower parts $\varphi_C(t) = \sum_{i=0}^{g-2} c_i t^i \in \mathbb{Q}[t]$, $\varphi_{C^\perp}(t) = \sum_{i=0}^{g^\perp-2} c_i^\perp t^i \in \mathbb{Q}[t]$ of Duursma's reduced polynomials $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i \in \mathbb{Q}[t]$ and the number $c_{g-1} = c_{g^\perp-1}^\perp \in \mathbb{Q}$ turn to determine completely $D_C(t), D_{C^\perp}(t)$. The final fifth section discusses some averaging, algebraic-geometric and probabilistic

interpretations of the coefficients $c_i \in \mathbb{Q}$ of Duursma's reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i \in \mathbb{Q}[t]$ of an additive code $(C, +) < (G^n, +)$, with a specific emphasis on the case of an \mathbb{F}_q -linear code $C \subset \mathbb{F}_q^n$, when C is not only a subgroup of $(\mathbb{F}_q^n, +)$ but also a subset with an \mathbb{F}_q^* -action $\mathbb{F}_q^* \times C \rightarrow C$, $(\lambda, a) \mapsto (\lambda a_1, \dots, \lambda a_n)$, preserving the Hamming weight. In general, $(|G| - 1)c_i$ with $0 \leq i \leq g - 1$ is shown to be the average coordinates of an intersection of $C \setminus \{0_G^n\}$ with $n - d - i$ coordinate hyperplanes in $(G^n, +)$. In the case of an \mathbb{F}_q -linear code $C \subset \mathbb{F}_q^n$, the presence of an algebro-geometric realization $C = \mathcal{E}_D \mathcal{L}_X(G)$, established by Pellikaan, Shen and van Wee in [17] allows to interpret the projectivization $\mathbb{P}(C)$ as an $\mathcal{L}_X(G - D)$ -orbit space of an explicit finite set of effective divisors of $\mathbb{F}_q(X)$. As a result, the coefficients of $\zeta_C(t) = \sum_{i=0}^{\infty} \mathcal{A}_i(C) t^i$ for $0 \leq i \leq g - 1$ are shown to be average cardinalities of appropriate $\mathcal{L}_X(G - D)$ -orbit spaces of effective divisors of $\mathbb{F}_q(X)$. In particular, $\binom{n}{d+i} \mathcal{A}_i(C) \in \mathbb{Z}^{\geq 0}$ for $\forall 0 \leq i \leq g + g^\perp - 2$ and $\mathcal{A}_i(C) \in \mathbb{Z}^{\geq 0}$ for $\forall i > g + g^\perp - 2$. As a result, Tsfasman-Vlăduț-Nogin's coefficients B_{d+i} , $0 \leq i \leq g - 1$ from $\mathcal{W}_C(x, y) = x^n + \sum_{i=0}^{n-d} B_i(x - y)^i y^{n-i}$, given in [19], turn to give the same information as the coefficients c_i of Duursma's reduced polynomial $D_C(t)$, due to $B_{d+i} = \binom{n}{d+i} (q - 1) c_i$ for $0 \leq i \leq g - 1$. We express c_i with $0 \leq i \leq g - 1$ by the probabilities of $a \in G^n$ of weight $d \leq \text{wt}(a) = s \leq d + i$ to belong to C . Similarly, c_i with $g \leq i \leq g + g^\perp - 2$ are related to the probabilities of $\pi \in \widehat{G}^n$ of weight $d^\perp \leq \text{wt}(\pi) = s \leq n - d - i$ to belong to C^\perp . Finally, the sum of the probabilities $\overline{p}_a^{(d+i)}$ of a $(d + i)$ -tuple of indices to contain the support of some $a \in C \setminus \{0_G^n\}$ is shown to be $(|G| - 1)c_i$ for $0 \leq i \leq g - 1$, while the sum of the probabilities $\overline{p}_\pi^{(n-d-i)}$ of an $(n - d - i)$ -tuple of indices to contain $\pi \in C^\perp \setminus \{\varepsilon\}$ turns to be $(|G| - 1)c_i |G|^{g-i+1}$ for $g \leq i \leq g^\perp - 2$. In the case of \mathbb{F}_q -linear codes, the factor $|G| - 1$ disappears by replacing $a \in C \setminus \{0_{\mathbb{F}_q}^n\}$ with $[a] \in \mathbb{P}(C) \subset \mathbb{P}(\mathbb{F}_q^n)$ and $\pi \in C^\perp \setminus \{0_{\mathbb{F}_q}^n\}$ with $[\pi] \in \mathbb{P}(C^\perp) \subset \mathbb{P}(\mathbb{F}_q^n)$.

2 Additive MDS weight enumerators

If $(G, +) \neq \{0_G\}$ is an additively written non-zero finite abelian group then the homomorphisms $\pi : (G, +) \rightarrow (\mathbb{C}^*, \cdot)$ are called multiplicative characters of G . If G is of order m then $\pi(G)$ consists of m -th roots of unity and, in particular, π maps to the subgroup (S^1, \cdot) of (\mathbb{C}^*, \cdot) , supported by the unit circle $S^1 := \{z \in \mathbb{C} \mid z\overline{z} = 1\}$. The set \widehat{G} of the multiplicative characters of G is a group with respect to the pointwise multiplication

$$\chi\pi : G \longrightarrow S^1, \quad (\chi\pi)(g) := \chi(g)\pi(g) \quad \text{for } \forall g \in G, \quad \forall \chi, \pi \in \widehat{G}.$$

The neutral element of this group is the trivial character

$$\varepsilon : G \longrightarrow \{1\}, \quad \varepsilon(g) = 1 \quad \text{for } \forall g \in G.$$

For an arbitrary $n \in \mathbb{N}$, the subgroups $(C, +)$ of $(G^n, +)$ are called additive codes of length n . With respect to the Hamming weight on G^n , defined by (1) and (3), there is a unique word $0_G^n \in G^n$ of weight 0, which belongs to any additive code $(C, +) < (G^n, +)$.

The minimal non-zero weight

$$d := \min\{\text{wt}(c) \in \mathbb{N} \mid c \in C \setminus \{0_G^n\}\}$$

of a word of $C \neq \{0_G^n\}$ is called the minimum weight of C or the minimum distance of C . If $C = \{0_G^n\}$ is the zero code, we assume that $d = 0$. As far as the Hamming distance

$$d : C \times C \longrightarrow \mathbb{Z}^{\geq 0}, \quad d(a, b) := \text{wt}(a - b)$$

is a metric, the decoding of an additive code of minimum weight $d \in \mathbb{N}$ with at most $\lfloor \frac{d-1}{2} \rfloor$ perturbed symbols is unique.

Here is a simple lemma on the puncturing and shortening of additive codes and their duals. The puncturing $\Pi_i : (C, +) \rightarrow (G^{n-1}, +)$ of an additive code $(C, +) \leq (G^n, +)$ at the i -th component is the group homomorphism, deleting the i -th component of each word $c \in C$, $\Pi_i(c_1, \dots, c_n) = (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n)$. By its very definition, the shortening $S_i : (C, +) \rightarrow (G^{n-1}, +)$ at the i -th component does not act on $c \in C$ with $c_i \neq 0$ and reduces to the puncturing Π_i on the words $c \in C$ with $c_i = 0$. The statement and the proof are the same as for \mathbb{F}_q -linear codes, as exposed in [9]. We give the proof for completeness.

Lemma 1. *Let $(C, +) \leq (G^n, +)$ be an additive code with dual $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$. Denote by $S_i : C \rightarrow G^{n-1}$, respectively, $S_i : C^\perp \rightarrow \widehat{G}^{n-1}$ the shortenings at the component, labeled by some $1 \leq i \leq n$ and put $\Pi_i : (C, +) \rightarrow (G^{n-1}, +)$, respectively, $\Pi_i : (C^\perp, \cdot) \rightarrow (\widehat{G}^{n-1}, \cdot)$ for the puncturings at i . Then*

$$S_i(C^\perp) = \Pi_i(C)^\perp \quad \text{and} \quad \Pi_i(C^\perp) = S_i(C)^\perp. \quad (6)$$

Proof. Towards the inclusion $S_i(C^\perp) \subseteq \Pi_i(C)^\perp$, it suffices to note that for an arbitrary $\pi \in C^\perp$ with $\pi_i = \varepsilon$ and an arbitrary $c \in C$, one has

$$S_i(\pi)(\Pi_i(c)) = \prod_{j \neq i} \pi_j(c_j) = \left[\prod_{j \neq i} \pi_j(c_j) \right] \pi_i(c_i) = \prod_{j=1}^n \pi_j(c_j) = \pi(c) = 1.$$

For the opposite inclusion $\Pi_i(C)^\perp \subseteq S_i(C^\perp)$, let us choose $\chi \in (\Pi_i(C)^\perp, \cdot) \leq (\widehat{G}^{n-1}, \cdot)$ and extend it to $\pi \in \widehat{G}^n$ with $\pi_j := \chi_j$ for $1 \leq j \leq i-1$, $\pi_i := \varepsilon$ and $\pi_j := \chi_{j-1}$ for $i+1 \leq j \leq n$. Then $\pi \in C^\perp$, according to

$$1 = \chi(\Pi_i(c)) = \left[\prod_{j=1}^{i-1} \chi_j(c_j) \right] \left[\prod_{j=i+1}^n \chi_{j-1}(c_j) \right] = \left[\prod_{j \neq i} \pi_j(c_j) \right] \varepsilon(c_i) = \pi(c),$$

for $\forall c \in C$. Thus, $\chi = S_i(\pi) \in S_i(C^\perp)$ and $\Pi_i(C^\perp) \subseteq S_i(C^\perp)$. That justifies the coincidence $S_i(C^\perp) = \Pi_i(C)^\perp$.

In order to check that $\Pi_i(C^\perp) \subseteq S_i(C)^\perp$, let $\pi \in C^\perp$ and $c \in C$ with $c_i = 0_G$. Then

$$\Pi_i(\pi)(S_i(c)) = \prod_{j \neq i} \pi_j(c_j) = \left[\prod_{j \neq i} \pi_j(c_j) \right] \pi_i(0_G) = \prod_{j=1}^n \pi_j(c_j) = \pi(c) = 1$$

reveals that $\Pi_i(\pi) \in S_i(C)^\perp$. Towards the coincidence $\Pi_i(C^\perp) = S_i(C)^\perp$, let us note that the application of $S_i(C^\perp) = \Pi_i(C)^\perp$ to C^\perp provides $S_i(C) \simeq S_i((C^\perp)^\perp) = \Pi_i(C^\perp)^\perp$, after combining with the natural group isomorphism $C \simeq (C^\perp)^\perp$. Taking the duals of both sides, one concludes that $S_i(C)^\perp \simeq [\Pi_i(C^\perp)^\perp]^\perp \simeq \Pi_i(C^\perp)$, whereas $|S_i(C)^\perp| = |\Pi_i(C^\perp)|$ and $\Pi_i(C^\perp) = S_i(C)^\perp$.

□

The next elementary lemma defines the genus of an additive code and reminds the Singleton Bound for additive codes. The proof is elementary and coincides with the one for the Singleton Bound of linear codes over finite fields. We provide it for completeness, as far as we have not found an available reference on it.

Lemma-Definition 2. *Any additive code $(C, +) \leq (G^n, +)$ of minimum distance $d \in \mathbb{Z}^{\geq 0}$ over a non-trivial finite abelian group $G \neq 0_G$ has non-negative genus*

$$g := n + 1 - d - \log_{|G|}(|C|).$$

Proof. Let us put $q := |G|$ and note that the subgroup $(G' := G^{d-1} \times 0_G^{n-d+1}, +) < (G^n, +)$ has trivial intersection $C \cap G' = \{0_G^n\}$ with C . If $G'' := G' + C$ is the subgroup of $(G^n, +)$, generated by G' and C then any element of the quotient group G''/G' admits a representative from $0_G^{d-1} \times G^{n-d+1}$. In particular, G''/G' is of order $|G''/G'| \leq |0_G^{d-1} \times G^{n-d+1}| = q^{n-d+1}$. The natural projection

$$\varphi : (C, +) \longrightarrow (G''/G', +), \quad \varphi(c) := c + G' \quad \text{for } \forall c \in C$$

is a group homomorphism with kernel $\ker \varphi = C \cap G' = \{0_G^n\}$. Therefore φ is injective and

$$|C| = |\varphi(C)| \leq |G''/G'| \leq q^{n-d+1}.$$

The real logarithmic function with base $q = |G| > 1$ is increasing, so that $\log_q(|C|) \leq n - d + 1$ and $g \geq 0$.

□

Among the additive codes $(C, +) \leq (G^n, +)$ over G of length n and cardinality $k = \log_{|G|}(|C|)$, the code C_o of genus $g = 0$ has unique decoding up to maximal possible number $\lfloor \frac{n+1-k}{2} \rfloor$ of perturbed symbols, we say that C_o is maximum distance separable or an additive MDS-code and denote it by $\text{MDS}(n, n+1-k)$. Here is another trivial result, which will be used in the sequel.

Lemma 3. *If $(C, +) \leq (G^n, +)$ is a non-trivial additive MDS-code of cardinality $|C| = |G|^k$ for some $k \in \mathbb{R}$, $k < n$ then the dual code $C^\perp := \{\pi \in \widehat{G}^n \mid \pi(c) = 1, \forall c \in C\} \simeq \widehat{G^n/C}$ of cardinality $|C^\perp| = \frac{|G|^n}{|C|} = |G|^{n-k}$ is $\text{MDS}(n, k+1)$.*

Proof. Note that an arbitrary character $\pi \in C^\perp \subset \widehat{G}^n$ provides a correctly defined homomorphism

$$\bar{\pi} : (G^n/C, +) \longrightarrow (S^1, \cdot), \quad \bar{\pi}(a + C) := \pi(a) \quad \text{for } \forall a \in G^n,$$

according to $\pi(a+c) = \pi(a)\pi(c) = \pi(a)$ for $\forall c \in C \leq \ker \pi$. Conversely, any character $\bar{\pi} : (G^n/C, +) \rightarrow (S^1, \cdot)$ of the quotient group $(G^n/C, +) = (G^n, +)/(C, +)$ lifts to a character $\pi : (G^n, +) \rightarrow (S^1, \cdot)$, $\pi(a) := \bar{\pi}(a+C)$ for $\forall a \in G^n$ with $C \leq \ker \pi$. Therefore $\pi \in C^\perp$ and there is a group isomorphism $(C^\perp, \cdot) \simeq (\widehat{G^n/C}, \cdot)$. In particular, the cardinality $|C^\perp| = |G^n/C| = [G^n : C] = \frac{|G^n|}{|C|} = |G|^{n-k}$.

The assumption $C \neq G^n$ implies that $C^\perp \neq \{\varepsilon\}$ has minimum distance $d^\perp \in \mathbb{N}$. Note that $k := \log_{|G|}(|C|) \in \mathbb{R}$ is a real number, $k < n$ and assume that the genus $g^\perp := n+1-d^\perp - \log_{|G|}(|C^\perp|) = n+1-d^\perp - (n-k) = k+1-d^\perp > 0$ of C^\perp is strictly positive, Then $d^\perp \leq k$ and for any $\pi \in C^\perp$ of weight $\text{wt}(\pi) = d^\perp$ there exists a k -tuple of indices $\alpha \in \binom{[n]}{k}$ with $\text{Supp}(\pi) \subseteq \alpha$. Let $\beta = \neg\alpha := \{1, \dots, n\} \setminus \alpha$ be the complement of α and

$$\Pi_\beta : C \longrightarrow \Pi_\beta(C) \subseteq G^k$$

be the puncturing at β . Note that Π_β is a homomorphism of additive groups with $\ker(\Pi_\beta) \cap C \neq \{0_G^n\}$. Otherwise, the restriction of Π_β on C is injective and $|G|^k = |C| = |\Pi_\beta(C)| \leq |G|^k$ implies that $\Pi_\beta(C) = G^k$. However, π has trivial components ε , labeled by β and for $\forall c \in C$ there holds

$$1 = \pi(c) = \prod_{i=1}^n \pi_i(c_i) = \prod_{i \in \alpha} \pi_i(c_i) = \Pi_\beta(\pi)(\Pi_\beta(c)).$$

Thus, $\Pi_\beta(\pi) \in \Pi_\beta(C)^\perp = (G^k)^\perp = \{\varepsilon^k\}$, whereas $\pi = \varepsilon^n$, contrary to the choice of $\pi \in C^\perp$ with $\text{wt}(\pi) = d^\perp \in \mathbb{N}$. That justifies $\ker \Pi_\beta \cap C \neq 0_G^n$. However, any word $c \in (\ker \Pi_\beta \cap C) \setminus \{0_G^n\}$ has support $\text{Supp}(c) \subseteq \beta$ and, therefore, is of weight $1 \leq \text{wt}(c) \leq n-k$. By assumption, C is of genus $g = n+1-d-k = 0$ or of minimum weight $d = n-k+1$ and does not contain non-zero words of weight $\leq n-k$. The contradiction justifies that C^\perp is of genus 0 or a Maximum Distance Separable code $\text{MDS}(n, k+1)$. □

In order to compute explicitly the weight distribution of an additive MDS-code $(\text{MDS}(n, d), +) < (G^n, +)$ of minimum distance d , one need one more lemma.

Lemma 4. *Let $(C, +) := \text{MDS}(n, d) \neq \{0_G^n\}$ be a non-zero additive MDS-code of length n and minimum distance $d \in \mathbb{N}$, over a finite abelian group G . Then:*

- (i) *in the case of $C \neq G^n$, for any $(d-1)$ -tuple of indices $\beta \in \binom{[n]}{d-1}$ the puncturing $\Pi_\beta : (C, +) \rightarrow (G^{n+1-d}, +)$ is a group isomorphism onto $(G^{n+1-d}, +)$;*
- (ii) *for any $\gamma \in \binom{[n]}{d}$ there are exactly $|G| - 1$ words of C with support γ ;*
- (iii) *C has $\mathcal{M}_{n,d}^{(d)} = \binom{n}{d}(|G| - 1)$ words of weight d .*

Proof. (i) If $|C| < |G|^n$ then the additive MDS-code C is of minimum distance $d = n+1 - \log_{|G|}(|C|) > 1$ and the puncturing $\Pi_\beta : (G^n, +) \rightarrow (G^{n+1-d}, +)$ is non-trivial, i.e., non-identical. Since C is of minimum distance d , the kernel $\ker \Pi_\beta = \{c \in G^n \mid \text{Supp}(c) \subseteq \beta\}$ of Π_β intersects C at the origin 0_G^n alone and $\Pi_\beta : C \rightarrow \Pi_\beta(C)$ is bijective. Therefore $|\Pi_\beta(C)| = |C| = |G|^{n+1-d} = |G^{n+1-d}|$ and $\Pi_\beta(C) = G^{n+1-d}$. If $c, c' \in \Pi_\beta^{-1}(a)$ for some

$a \in G^{n+1-d}$ then $\text{Supp}(c - c') \subseteq \beta \in \binom{[n]}{d-1}$ and $c = c'$. In such a way, the puncturing $\Pi_\beta : (C, +) \rightarrow (G^{n+1-d}, +)$ is shown to be a group isomorphism.

(ii) The additive code $C = G^n$ is of minimum distance $d = 1$ and for any $\gamma \in \binom{[n]}{1}$ there are exactly $|G| - 1$ words $(0_G^{\gamma-1}, g, 0_G^{n-\gamma}) \in G^n$, $g \in G \setminus \{0_G\}$ with support γ . From now on, we assume that $C \subsetneq G^n$ is a proper subgroup of $(G^n, +)$. If $\gamma \in \binom{[n]}{d}$ and $i \in \gamma$ let $\beta := \gamma \setminus \{i\} \in \binom{[n]}{d-1}$, $\delta := \neg\gamma = \{1, \dots, n\} \setminus \gamma$ and recall from (i) that the puncturing $\Pi_\beta : (C, +) \rightarrow (G^{n+1-d}, +)$ is a group isomorphism. In particular, for any $a \in G^{n+1-d}$ with $a_\delta = 0_G^{n-d}$ and an arbitrary $a_i \in G \setminus \{0_G\}$ there is a unique $c \in C$ with $\Pi_\beta(c) = a$. Therefore, the support $\text{Supp}(c) \subseteq \beta \cup \{i\} = \gamma$ of c is contained in γ and since C is of minimum distance d , there follows $\text{Supp}(c) = \gamma$. In such a way, we have shown the existence of at least $|G| - 1$ words of C with support γ . Since any $c \in C$ with $\text{Supp}(c) = \gamma$ is among the constructed ones, there are exactly $|G| - 1$ words $c \in C$ with $\text{Supp}(c) = \gamma$.

(iii) is an immediate consequence of (ii) and the fact that the number of the d -tuples $\gamma \in \binom{[n]}{d}$ is $\binom{n}{d}$. □

The next proposition computes the homogeneous weight enumerator $\mathcal{M}_{n,d}(x, y)$ of an additive MDS-code $\text{MDS}(n, d)$ of length n and minimum distance d over an arbitrary finite abelian group G . That allows to express the homogeneous weight enumerator of an arbitrary additive code $(C, +) \leq (G^n, +)$ of minimum distance d with dual $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ of minimum distance d^\perp as a \mathbb{Q} -linear combination of the polynomials $\mathcal{M}_{n,d}(x, y)$, $\mathcal{M}_{n,d+1}(x, y)$, \dots , $\mathcal{M}_{n,n+2-d^\perp}(x, y)$ (cf. Proposition-Definition 7 from the next section).

Proposition 5. *An arbitrary additive MDS-code $(\text{MDS}(n, d), +) \leq (G^n, +)$ of minimum distance $d \in \mathbb{N}$ has*

$$\mathcal{M}_{n,d}^{(s)} = \binom{n}{s} (|G| - 1) \left[\sum_{i=0}^{s-d} (-1)^i \binom{s-1}{i} |G|^{s-d-i} \right] \quad (7)$$

words of weight s for all $d \leq s \leq n$.

Proof. Let $q := |G|$ be the order of G . If $\text{MDS}(n, d) = G^n$ then $d = 1$ and for any $1 \leq s \leq n$ there are $\binom{n}{s}$ subsets $\gamma \subseteq \{1, \dots, n\}$ of cardinality $|\gamma| = s$. For any such γ there are $(q - 1)^s$ words $(a_\gamma, a_{-\gamma} = 0_G^{n-s}) \in G^n$, $a_{\gamma_1}, \dots, a_{\gamma_s} \in G \setminus \{0_G\}$ with support γ . Therefore $\mathcal{M}_{n,1}^{(s)} = \binom{n}{s} (q - 1)^s$ for all $1 \leq s \leq n$. Bearing in mind that

$$\begin{aligned} \binom{n}{s} (q - 1) \left[\sum_{i=0}^{s-d} (-1)^i \binom{s-1}{i} q^{s-d-i} \right] &= \binom{n}{s} (q - 1) \left[\sum_{i=0}^{s-1} (-1)^i \binom{s-1}{i} q^{s-1-i} \right] = \\ &= \binom{n}{s} (q - 1) (q - 1)^{s-1} = \binom{n}{s} (q - 1)^s, \end{aligned}$$

one proves (7) for $\text{MDS}(n, d) = \text{MDS}(n, 1) = G^n$.

From now on, assume that the additive MDS-code $(\text{MDS}(n, d), +) \subsetneq (G^n, +)$ is a proper subgroup of $(G^n, +)$ and note that the dual $\text{MDS}(n, d)^\perp \leq (\widehat{G}^n, \cdot)$ of cardinality

$|\text{MDS}(n, d)^\perp| = \frac{q^n}{|\text{MDS}(n, d)|} = \frac{q^n}{q^{n+1-d}} = q^{d-1}$ is of genus 0 by Lemma 3. Therefore $d^\perp = n + 2 - d$ and

$$\text{MDS}(n, d)^\perp = \text{MDS}(n, n + 2 - d). \quad (8)$$

In particular, $d^\perp \geq 2$ by $d \leq n$. We note that

$$\mu_d^{(s)} := (q-1) \left[\sum_{i=0}^{s-d} (-1)^i \binom{s-1}{i} q^{s-d-i} \right] \quad \text{for } \forall d \leq s \leq n$$

is independent of n and show that $\mathcal{M}_{n,d}^{(s)} = \binom{n}{s} \mu_d^{(s)}$ by an induction on the length n . If $n = 1$ then the assumption $|\text{MDS}(1, d)| = q^{2-d} < q = |G|$ requires $d > 1$, which is an absurd. Thus, the only additive MDS-code of length 1 is $\text{MDS}(1, 1) = G$ and (7) is true for all additive MDS-codes of length 1. Assume that (7) holds for all additive MDS-codes of length $n-1$, put $C := \text{MDS}(n, d)$ and consider the shortening $S_i : C \rightarrow S_i(C)$. By its very definition, the shortening S_i does not erase non-zero components and preserves the minimum distance d . On the other hand, the puncturing $\Pi_i : C^\perp \rightarrow \Pi_i(C^\perp)$ of the dual code $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ is bijective, as far as $\pi \in \ker \Pi_i \cap C^\perp$ exactly when $\text{Supp}(\pi) \subseteq \{i\}$ and $d^\perp \geq 2$. Making use of $\Pi_i(C^\perp) = S_i(C)^\perp$, established by (6) from Lemma 1, one concludes that

$$|S_i(C)^\perp| = |\Pi_i(C^\perp)| = |C^\perp| = q^{d-1}$$

for $(S_i(C)^\perp, \cdot) \leq (\widehat{G}^{n-1}, \cdot)$, whereas $|S_i(C)| = \frac{q^{n-1}}{|S_i(C)^\perp|} = q^{n-d}$. Thus, $S_i(C)$ is of genus $g(S_i(C)) := (n-1) + 1 - d(S_i(C)) - \log_q |S_i(C)| = 0$ and by the inductual hypothesis, the homogeneous weight enumerator $\mathcal{W}_{S_i(C)}(x, y) := x^{n-1} + \sum_{s=d}^{n-1} \mathcal{W}_{S_i(C)}^{(s)} x^{n-1-s} y^s$ of $S_i(C)$ has coefficients $\mathcal{W}_{S_i(C)}^{(s)} = \binom{n-1}{s} \mu_d^{(s)}$ for $\forall d \leq s \leq n-1$. Let

$$S := \sum_{i=1}^n S_i : \mathcal{M}_{n,d}(x, y) \longrightarrow S\mathcal{M}_{n,d}(x, y) = \sum_{i=1}^n S_i \mathcal{M}_{n,d}(x, y) := \sum_{i=1}^n \mathcal{W}_{S_i(C)}(x, y)$$

be the map, which transforms the homogeneous weight enumerator $\mathcal{M}_{n,d}(x, y)$ of $C = \text{MDS}(n, d)$ into the sum of the homogeneous weight enumerator of $S_i(C)$. By the inductual hypothesis,

$$S\mathcal{M}_{n,d}(x, y) = nx^{n-1} + \sum_{s=d}^{n-1} n \binom{n-1}{s} \mu_d^{(s)} x^{n-1-s} y^s. \quad (9)$$

On the other hand, an arbitrary monomial $x^{n-s} y^s$ of $\mathcal{M}_{n,d}(x, y)$ counts a word $c \in C$ with support $\text{Supp}(c) = \gamma = \{\gamma_1, \dots, \gamma_s\} \in \binom{[n]}{s}$. The shortenings S_{γ_j} with $1 \leq j \leq s$ do not produce a word from $\prod_{i=1}^n S_i(C)$, while the shortenings S_i at $i \in \{1, \dots, n\} \setminus \gamma$ yield a word of $S_i(C)$. Thus, $x^{n-s} y^s$ is transformed into $(n-s)x^{n-1-s} y^s = \frac{\partial}{\partial x}(x^{n-s} y^s)$. Since the partial derivative

$$\frac{\partial}{\partial x} : \mathbb{C}[x, y]^{(n)} \longrightarrow \mathbb{C}[x, y]^{(n-1)}$$

is a \mathbb{C} -linear map of the homogeneous polynomials $\mathbb{C}[x, y]^{(n)}$ of x, y of degree n in the homogeneous polynomials $\mathbb{C}[x, y]^{(n-1)}$ of degree $n - 1$, the polynomial

$$S\mathcal{M}_{n,d}(x, y) = \frac{\partial}{\partial x}\mathcal{M}_{n,d}(x, y).$$

Combining with (9) and comparing the coefficients of $x^{n-1-s}y^s$ for $\forall d \leq s \leq n$, one concludes that $(n-s)\mathcal{M}_{n,d}^{(s)} = n\binom{n-1}{s}\mu_d^{(s)}$. That completes the proof of

$$\mathcal{M}_{n,d}^{(s)} = \frac{n\binom{n-1}{s}}{n-s}\mu_d^{(s)} = \binom{n}{s}\mu_d^{(s)} \quad \text{for } \forall 1 \leq s \leq d.$$

□

The following corollary will be useful for expressing Mac Williams identities for an arbitrary pair C, C^\perp of mutually dual additive codes in terms of their ζ -polynomials $P_C(t), P_{C^\perp}(t) \in \mathbb{Q}[t]$. In order to formulate and prove it, we consider the set $\mathbb{C}[x, y]^{(n)}[[t]]$ of the formal power series of t , whose coefficients are homogeneous polynomials of x, y of degree n . For arbitrary $\eta(x, y, t) \in \mathbb{C}[x, y]^{(n)}[[t]]$ and $s \in \mathbb{Z}^{\geq 0}$, let us denote by $\text{Coeff}_{t^s}(\eta(x, y, t)) \in \mathbb{C}[x, y]^{(n)}$ the coefficient of t^s from $\eta(x, y, t)$. The proof of the proposition coincides with the one of Proposition 1 from Duursma's [5] on \mathbb{F}_q -linear MDS-codes.

Corollary 6. *The homogeneous weight enumerator $\mathcal{M}_{n,d}(x, y)$ of an additive MDS-code $(\text{MDS}(n, d), +) \leq (G^n, +)$ of minimum distance d is uniquely determined by the equality of polynomials*

$$\frac{\mathcal{M}_{n,d}(x, y) - x^n}{|G| - 1} = \text{Coeff}_{t^{n-d}} \left(\frac{[xt + y(1-t)]^n}{(1-t)(1-|G|t)} \right). \quad (10)$$

Proof. Let us denote $q := |G|$, $\eta(x, y, t) := \frac{[xt + y(1-t)]^n}{(1-t)(1-qt)}$ and note that

$$\eta(x, y, t) = \sum_{s=0}^n \binom{n}{s} \frac{t^{n-s}(1-t)^s}{(1-t)(1-qt)} x^{n-s}y^s$$

has coefficient

$$\text{Coeff}_{t^{n-d}}\eta(x, y, t) = \sum_{s=0}^n \binom{n}{s} \text{Coeff}_{t^{s-d}} \left(\frac{(1-t)^{s-1}}{1-qt} \right) x^{n-s}y^s$$

of t^{n-d} . Since $\frac{(1-t)^{s-1}}{1-qt} \in \mathbb{C}[[t]]$ has no pole at $t = 0$, one has $\text{Coeff}_{t^{s-d}} \left(\frac{(1-t)^{s-1}}{1-qt} \right) = 0$ for $\forall 0 \leq s \leq d-1$ and

$$\text{Coeff}_{t^{n-d}}\eta(x, y, t) = \sum_{s=d}^n \binom{n}{s} \text{Coeff}_{t^{s-d}} \left(\frac{(1-t)^{s-1}}{1-qt} \right) x^{n-s}y^s.$$

Making use of (7) from Proposition 5, one reduces the proof of (10) to

$$\begin{aligned} \text{Coeff}_{t^{s-d}} \left(\frac{(1-t)^{s-1}}{1-qt} \right) &= \text{Coeff}_{t^{s-d}} \left(\left[\sum_{i=0}^{s-1} \binom{s-1}{i} (-1)^i t^i \right] \left(\sum_{j=0}^{\infty} q^j t^j \right) \right) = \\ &= \sum_{i=0}^{s-d} \binom{s-1}{i} (-1)^i q^{s-d-i} \quad \text{for } \forall d \leq s \leq n. \end{aligned}$$

□

3 Mac Williams identities as a functional equation of Duursma's reduced polynomials

The next proposition reminds Duursma's definition of a ζ -polynomial $P_C(t)$ of an \mathbb{F}_q -linear code $C \subset \mathbb{F}_q^n$ and expresses Mac Williams identities for the weight distribution of C , $C^\perp \subset \mathbb{F}_q^n$ as a functional equation on $P_C(t)$, $P_{C^\perp}(t)$. All properties of C , C^\perp , which are used by Duursma's construction hold for additive codes $(C, +) \leq (G^n, +)$ and their duals $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$. We formulate in terms of additive codes and provide the proofs for completeness.

Proposition-Definition 7. *Let $(C, +) \leq (G^n, +)$ be an additive code of genus g and minimum distance $d \geq 2$ with dual $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ of genus g^\perp and minimum distance $d^\perp \geq 2$. Then there exist unique polynomials*

$$P_C(t) = \sum_{i=0}^{g+g^\perp} a_i t^i, \quad P_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp} a_i^\perp t^i \in \mathbb{Q}[t]$$

with $P_C(1) = P_{C^\perp}(1) = 1$, whose coefficients express the homogeneous weight enumerators

$$\mathcal{W}_C(x, y) = \sum_{i=0}^{g+g^\perp} a_i \mathcal{M}_{n, d+i}(x, y), \quad (11)$$

$$\mathcal{W}_{C^\perp}(x, y) = \sum_{i=0}^{g+g^\perp} a_i^\perp \mathcal{M}_{n, d^\perp+i}(x, y) \quad (12)$$

by appropriate MDS weight enumerators. Mac Williams identities for C , C^\perp are equivalent to the functional equation

$$P_{C^\perp}(t) = P_C \left(\frac{1}{|G|t} \right) |G|^g t^{g+g^\perp} \quad (13)$$

for $P_C(t)$, $P_{C^\perp}(t)$. From now on, $P_C(t)$, $P_{C^\perp}(t)$ are referred to as the ζ -polynomials of C , C^\perp and the rational functions

$$\zeta_C(t) := \frac{P_C(t)}{(1-t)(1-|G|t)}, \quad \zeta_{C^\perp}(t) := \frac{P_{C^\perp}(t)}{(1-t)(1-|G|t)} \quad (14)$$

are called the ζ -functions of C , C^\perp .

Proof. For arbitrary $n, d \in \mathbb{N}$, $d \leq n$ let $\mathbb{Q}[x, y]_{\geq d}^{(n)}$ be the \mathbb{Q} -linear space of the homogeneous polynomials of x, y of degree n , whose monomials with non-zero coefficients are of degree $\geq d$ with respect to y . Then $\{x^{n-i}y^i \mid d \leq i \leq n\}$ is a \mathbb{Q} -basis of $\mathbb{Q}[x, y]_{\geq d}^{(n)}$, as well as any set of polynomials $f_i(x, y) \in \mathbb{Q}[x, y]_{\geq d}^{(n)}$, $d \leq i \leq n$ with $\text{Coeff}_{x^{n-i}y^i}(f_i(x, y)) \neq 0$ for all $d \leq i \leq n$. In particular, for $\forall 0 \leq i \leq n - d$ the polynomials

$$\mathcal{M}_{n,d+i}(x, y) - x^n = \sum_{s=d+i}^n \mathcal{M}_{n,d+i}^{(s)} x^{n-s} y^s$$

with $\mathcal{M}_{n,d+i}^{(d+i)} = \binom{n}{d+i}(|G| - 1) > 0$ constitute a \mathbb{Q} -basis of $\mathbb{Q}[x, y]_{\geq d}^{(n)}$. The coordinates $a_0, \dots, a_{n-d} \in \mathbb{Q}$ of $\mathcal{W}_C(x, y) - x^n \in \mathbb{Q}[x, y]_{\geq d}^{(n)}$ with respect to $\mathcal{M}_{n,d}(x, y) - x^n, \dots, \mathcal{M}_{n,n}(x, y) - x^n$ provide a uniquely determined ζ -polynomials $P_C(t) = \sum_{i=0}^{n-d} a_i t^i \in \mathbb{Q}[t]$ with

$$\mathcal{W}_C(x, y) - x^n = \sum_{i=0}^{n-d} a_i [\mathcal{M}_{n,d+i}(x, y) - x^n]. \quad (15)$$

Similar considerations provide a unique ζ -polynomial $P_{C^\perp}(t) = \sum_{i=0}^{n-d^\perp} a_i^\perp t^i \in \mathbb{Q}[t]$ of $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ with

$$\mathcal{W}_{C^\perp}(x, y) - x^n = \sum_{i=0}^{n-d^\perp} a_i^\perp [\mathcal{M}_{n,d^\perp+i}(x, y) - x^n]. \quad (16)$$

According to Delsarte's [2] (cf. also Wood's [20]), Mac Williams identities for the weight distribution of an additive code $(C, +) \leq (G^n, +)$ and its dual $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ reads as

$$|C| \mathcal{W}_{C^\perp}(x, y) = \mathcal{W}_C(x + (|G| - 1)y, x - y). \quad (17)$$

By assumption $d \geq 2$, so that $(\text{MDS}(n, d), +) \not\leq (G^n, +)$ is a non-trivial MDS-code and its dual $\text{MDS}(n, d)^\perp = \text{MDS}(n, n + 2 - d)$ is also MDS by (8). Denoting $q := |G|$, one expresses Mac Williams identities (17) for $\text{MDS}(n, d)$ and $\text{MDS}(n, d)^\perp$ as

$$q^{n+1-d} \mathcal{M}_{n,n+2-d}(x, y) = \mathcal{M}_{n,d}(x + (q - 1)y, x - y). \quad (18)$$

If $k := \log_q(|C|)$ then substituting (15), (16) in (17) and using (18), one rewrites (17) in

the form

$$\begin{aligned}
& q^k \left\{ \sum_{i=0}^{n-d^\perp} a_i^\perp [\mathcal{M}_{n,d^\perp+i}(x,y) - x^n] + x^n \right\} = \\
& q^k [\mathcal{W}_{C^\perp}(x,y) - x^n] + q^k x^n = \\
& \{\mathcal{W}_C(x + (q-1)y, x-y) - [x + (q-1)y]^n\} + [x + (q-1)y]^n = \\
& \sum_{i=0}^{n-d} a_i \{\mathcal{M}_{n,d+i}(x + (q-1)y, x-y) - [x + (q-1)y]^n\} + [x + (q-1)y]^n = \\
& \sum_{i=0}^{n-d} a_i \left\{ q^{n+1-d-i} \mathcal{M}_{n,n+2-d-i}(x,y) - [x + (q-1)y]^n \right\} + [x + (q-1)y]^n = \\
& \sum_{i=0}^{n-d} a_i q^{n+1-d-i} [\mathcal{M}_{n,n+2-d-i}(x,y) - x^n] + \\
& + \left(\sum_{i=0}^{n-d} a_i q^{n+1-d-i} \right) x^n + \left(1 - \sum_{i=0}^{n-d} a_i \right) [x + (q-1)y]^n = \\
& \sum_{i=0}^{n-d} a_i q^{n+1-d-i} [\mathcal{M}_{n,n+2-d-i}(x,y) - x^n] + \\
& + q^{n+1-d} P_C \left(\frac{1}{q} \right) x^n + [1 - P_C(1)] [x + (q-1)y]^n.
\end{aligned}$$

In terms of the summation indices, which equal the minimum distances of the corresponding MDS weight enumerators, the above reads as

$$\begin{aligned}
& q^k \left\{ \sum_{j=d^\perp}^n a_{j-d^\perp}^\perp [\mathcal{M}_{n,j}(x,y) - x^n] \right\} + q^k x^n = \\
& \sum_{j=2}^{n+2-d} a_{n+2-d-j} q^{j-1} [\mathcal{M}_{n,j}(x,y) - x^n] + q^{n+1-d} P_C \left(\frac{1}{q} \right) x^n + [1 - P_C(1)] [x + (q-1)y]^n.
\end{aligned} \tag{19}$$

Bearing in mind that $d^\perp \geq 2$, one compares the coefficients of $x^{n-1}y$ from (19) and concludes that $P_C(1) = 1$. As a result, (15) reduces to (11) and (16) takes the form (12), due to the presence of a canonical isomorphism $(C^\perp)^\perp \simeq C$. Then the comparison of the coefficients of x^n yields $q^k = q^{n+1-d} P_C \left(\frac{1}{q} \right)$, whereas $P_C \left(\frac{1}{q} \right) = q^{-g} = \left(\frac{1}{q} \right)^g$ for the genus $g := n+1-d-k$ of C . The comparison of the coefficients of $\mathcal{M}_{n,j}(x,y) - x^n$ with $j < d^\perp$ or $j > n+2-d$ from (19) implies $a_i = a_i^\perp = 0$ for all $i > n+2-d-d^\perp = g+g^\perp$ and allows to write Mac Williams identities (19) for C , C^\perp in the form

$$q^k \sum_{j=d^\perp}^{n+2-d} a_{j-d^\perp}^\perp [\mathcal{M}_{n,j}(x,y) - x^n] = \sum_{j=d^\perp}^{n+2-d} q^{j-1} a_{n+2-d-j} [\mathcal{M}_{n,j}(x,y) - x^n],$$

which is equivalent to

$$q^k a_{j-d^\perp}^\perp = q^{j-1} a_{n+2-d-j} \quad \text{for } \forall d^\perp \leq j \leq n+2-d.$$

Under the substitution $i := j - d^\perp$, these amount to

$$a_i^\perp = q^{i-g^\perp} a_{g+g^\perp-i} \quad \text{for } \forall 0 \leq i \leq g+g^\perp. \quad (20)$$

If $P_C(t) := \sum_{i=0}^{g+g^\perp} a_i t^i$, $P_{C^\perp}(t) := \sum_{i=0}^{g+g^\perp} a_i^\perp t^i \in \mathbb{Q}[t]$ then one easily checks that the equalities of the coefficients of t^i , $0 \leq i \leq g+g^\perp$ from both sides of (13) can be expressed as (20). □

Corollary 8. *If $(C, +) \leq (G^n, +)$ is an additive code of minimum distance $d \geq 2$ with dual $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ of minimum distance $d^\perp \geq 2$ then the ζ -polynomials of C , C^\perp are the uniquely determined polynomials $P_C(t) = \sum_{i=0}^{g+g^\perp} a_i t^i \in \mathbb{Q}[t]$, respectively, $P_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp} a_i^\perp t^i \in \mathbb{Q}[t]$ of degree $\deg P_C(t) \leq g+g^\perp$, $\deg P_{C^\perp}(t) \leq g+g^\perp$ with $P_C(1) = P_{C^\perp}(1) = 1$, which satisfy the equalities*

$$\frac{\mathcal{W}_C(x, y) - x^n}{|G| - 1} = \text{Coeff}_{t^{n-d}} \left(P_C(t) \frac{[xt + y(1-t)]^n}{(1-t)(1-|G|t)} \right), \quad (21)$$

respectively,

$$\frac{\mathcal{W}_{C^\perp}(x, y) - x^n}{|G| - 1} = \text{Coeff}_{t^{n-d^\perp}} \left(P_{C^\perp}(t) \frac{[xt + y(1-t)]^n}{(1-t)(1-|G|t)} \right) \quad (22)$$

of homogeneous polynomials of x, y of degree n .

Proof. Let $q := |G|$ be the order of G . For an arbitrary polynomial $P_C(t) = \sum_{i \geq 0} a_i t^i \in \mathbb{Q}[t]$, whose degree will be bounded in the sequel, note that

$$\begin{aligned} \text{Coeff}_{t^{n-d}} \left(P_C(t) \frac{[xt + y(1-t)]^n}{(1-t)(1-qt)} \right) &= \text{Coeff}_{t^{n-d}} \left(\left[\sum_{i \geq 0} a_i t^i \right] \frac{[xt + y(1-t)]^n}{(1-t)(1-qt)} \right) = \\ &= \sum_{i \geq 0} a_i \text{Coeff}_{t^{n-d-i}} \left(\frac{[xt + y(1-t)]^n}{(1-t)(1-qt)} \right). \end{aligned}$$

The rational function $\frac{[xt+y(1-t)]^n}{(1-t)(1-qt)}$ without a pole at $t = 0$ has vanishing coefficients of t^{n-d-i} for $\forall i > n-d$, so that the polynomial $P_C(t) = \sum_{i=0}^{n-d} a_i t^i \in \mathbb{Q}[t]$ is of degree $\deg P_C(t) \leq n-d$. Now, (10) and (15) specify that

$$\text{Coeff}_{t^{n-d}} \left(P_C(t) \frac{[xt + y(1-t)]^n}{(1-t)(1-qt)} \right) = \sum_{i=0}^{n-d} a_i \left(\frac{\mathcal{M}_{n,d+i}(x,y) - x^n}{q-1} \right) = \frac{\mathcal{W}_C(x,y) - x^n}{q-1}.$$

Similar considerations provide (22) for $P_{C^\perp}(t) = \sum_{i=0}^{n-d^\perp} a_i^\perp t^i \in \mathbb{Q}[t]$. The remaining part of the proof of Proposition-Definition 7 establishes that $\deg P_C(t) \leq g + g^\perp$, $\deg P_{C^\perp}(t) \leq g + g^\perp$ and $P_C(1) = P_{C^\perp}(1) = 1$. \square

In order to get an impression of the (integral) denominators of the coefficients $a_i \in \mathbb{Q}$ of $P_C(t)$, let us plug in $\mathcal{M}_{n,d+i}(x,y) = x^n + \sum_{s=d+i}^n \mathcal{M}_{n,d+i}^{(s)} x^{n-s} y^s$ in (11) and exchange the summation order. That provides

$$x^n + \sum_{s=d}^{n+2-d^\perp} \left(\sum_{i=0}^{s-d} a_i \mathcal{M}_{n,d+i}^{(s)} \right) x^{n-s} y^s + \sum_{s=n+3-d^\perp}^n \left(\sum_{i=0}^{g+g^\perp} a_i \mathcal{M}_{n,d+i}^{(s)} \right) x^{n-s} y^s =$$

due to $P_C(1) = 1$. Comparing the coefficients of $x^{n-s} y^s$ for $\forall d \leq s \leq n+2-d^\perp$, one concludes that $a_i, \forall 0 \leq i \leq n+2-d^\perp-d = g+g^\perp$ constitute a solution of the linear system of equations

$$\begin{pmatrix} \mathcal{M}_{n,d}^{(d)} & \dots & 0 & \dots & 0 \\ \mathcal{M}_{n,d}^{(d+1)} & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{M}_{n,d}^{(s)} & \dots & \mathcal{M}_{n,s}^{(s)} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{M}_{n,d}^{(n+2-d^\perp)} & \dots & \mathcal{M}_{n,s}^{(n+2-d^\perp)} & \dots & \mathcal{M}_{n,n+2-d^\perp}^{(n+2-d^\perp)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_{s-d} \\ \dots \\ a_{g+g^\perp} \end{pmatrix} = \begin{pmatrix} \mathcal{W}_C^{(d)} \\ \mathcal{W}_C^{(d+1)} \\ \dots \\ \mathcal{W}_C^{(s)} \\ \dots \\ \mathcal{W}_C^{(g+g^\perp)} \end{pmatrix}$$

with lower triangular coefficient matrix and determinant

$$\Delta = \prod_{s=d}^{n+2-d^\perp} \mathcal{M}_{n,s}^{(s)} = (q-1)^{n+3-d^\perp-d} \prod_{s=d}^{n+2-d^\perp} \binom{n}{s} = (q-1)^{g+g^\perp+1} \prod_{j=0}^{g+g^\perp} \binom{n}{d+j} \neq 0$$

by (7). Cramer's rule applies to provide $a_i = \frac{\Delta_i}{\Delta}$ with $\Delta_i \in \mathbb{Z}$ for $\forall 0 \leq i \leq g+g^\perp$, so that

$$\Delta a_i = (q-1)^{g+g^\perp+1} \prod_{j=0}^{g+g^\perp} \binom{n}{d+j} a_i \in \mathbb{Z}.$$

In other words, the denominators of a_i are integral divisors of $(q-1)^{g+g^\perp+1} \prod_{j=0}^{g+g^\perp} \binom{n}{d+j}$ for $\forall 0 \leq i \leq g+g^\perp$.

In the proof of Proposition-Definition 7 we have established that the ζ -polynomial $P_C(t) = \sum_{i=0}^{g+g^\perp} a_i t^i \in \mathbb{Q}[t]$ of an additive code $(C, +) \leq (G^n, +)$ of genus g with dual $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ of genus g^\perp has values $P_C(1) = 1$ and $P_C\left(\frac{1}{|G|}\right) = \left(\frac{1}{|G|}\right)^g$ at the simple poles of the ζ -function $\zeta_C(t) = \frac{P_C(t)}{(1-t)(1-|G|t)}$, whenever $d \geq 2$ and $d^\perp \geq 2$. Therefore $P_C(t) - t^g \in \mathbb{Q}[t]$ vanishes at $t = 1$, $t = \frac{1}{|G|}$ and

$$D_C(t) := \frac{P_C(t) - t^g}{(1-t)(1-qt)} \in \mathbb{Q}[t]$$

is a polynomial of degree $\deg D_C(t) \leq g + g^\perp - 2$. If $g = g^\perp = 0$, then $P_C(t) \equiv P_{C^\perp}(t) \equiv 1$ are constant and the polynomials $D_C(t) \equiv D_{C^\perp}(t) \equiv 0$ vanish identically. In the case of an \mathbb{F}_q -linear code C , [12] refers to $D_C(t)$ as to Duursma's reduced polynomial of C and making use of (11) expresses the weight distribution of C by the means of the coefficients c_i of $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i \in \mathbb{Q}[t]$. The present note adopts another definition of $D_C(t)$ and by the means of (21) establishes its equivalence to the aforementioned one. Duursma's reduced polynomials $D_C(t), D_{C^\perp}(t) \in \mathbb{Q}[t]$ are used for expressing Mac Williams identities for $(C, +) < (G^n, +)$, $(C^\perp, \cdot) < (\widehat{G}^n, \cdot)$ as Polarized Riemann-Roch Conditions. Besides, (27) from the next Proposition 9 reveals that the denominators of the coefficients c_i of $D_C(t)$ are integral divisors of $(|G| - 1) \binom{n}{d+i}$ for $\forall 0 \leq i \leq g + g^\perp - 2$. In the case of \mathbb{F}_q -linear codes $C, C^\perp \subset \mathbb{F}_q^n$, Corollary 10 specifies that the denominators of c_i divide $\binom{n}{d+i}$ for $\forall 0 \leq i \leq g + g^\perp - 2$.

Proposition 9. (Compare with Proposition 1 from [12]) *Let $(C, +) \leq (G^n, +)$ be an additive code of cardinality $|G|^k$ and minimum distance $d \geq 2$ with dual $(C^\perp, \cdot) \leq (\widehat{G}^n, \cdot)$ of cardinality $|G|^{n-k}$ and minimum distance $d^\perp \geq 2$. Then there exist unique Duursma's reduced polynomials $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i \in \mathbb{Q}[t]$, respectively, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i \in \mathbb{Q}[t]$ of $\deg D_C(t) \leq g + g^\perp - 2$, $\deg D_{C^\perp}(t) \leq g + g^\perp - 2$, such that*

$$\mathcal{W}_C(x, y) = \mathcal{M}_{n, n+1-k}(x, y) + \sum_{i=0}^{g+g^\perp-2} \left[\binom{n}{d+i} (|G| - 1) c_i \right] (x - y)^{n-d-i} y^{d+i}, \quad (23)$$

respectively,

$$\mathcal{W}_{C^\perp}(x, y) = \mathcal{M}_{n, k+1}(x, y) + \sum_{i=0}^{g+g^\perp-2} \left[\binom{n}{d^\perp+i} (|G| - 1) c_i^\perp \right] (x - y)^{n-d^\perp-i} y^{d^\perp+i} \quad (24)$$

for the genus $g = n + 1 - d - k$ of C and the genus $g^\perp = k + 1 - d^\perp$ of C^\perp .

The polynomials $D_C(t), D_{C^\perp}(t)$ can be determined by the equalities

$$D_C(t) = \frac{P_C(t) - t^g}{(1-t)(1-|G|t)}, \quad \text{respectively,} \quad D_{C^\perp}(t) = \frac{P_{C^\perp}(t) - t^{g^\perp}}{(1-t)(1-|G|t)}. \quad (25)$$

Mac Williams identities for the weight distributions of C , C^\perp are equivalent to the functional equation

$$D_{C^\perp}(t) = D_C\left(\frac{1}{|G|t}\right) |G|^{g-1} t^{g+g^\perp-2} \quad (26)$$

of the corresponding Duursma's reduced polynomials and

$$(|G| - 1) \binom{n}{d+i} c_i = \sum_{s=0}^{d+i} \left(\mathcal{W}_C^{(s)} - \mathcal{M}_{n,n+1-k}^{(s)} \right) \binom{n-s}{n-d-i} \in \mathbb{Z}, \quad (27)$$

$$(|G| - 1) \binom{n}{d^\perp+i} c_i^\perp = \sum_{s=0}^{d^\perp+i} \left(\mathcal{W}_{C^\perp}^{(s)} - \mathcal{M}_{n,k+1}^{(s)} \right) \binom{n-s}{n-d^\perp-i} \in \mathbb{Z} \quad (28)$$

for $\forall 0 \leq i \leq g + g^\perp - 2$.

Proof. Towards the existence of $D_C(t) \in \mathbb{Q}[t]$, let $q := |G|$ be the order of G and

$$\widetilde{D}_C(t) := \frac{P_C(t) - t^g}{(1-t)(1-qt)} = \sum_{i=0}^{g+g^\perp-2} c_i t^i \in \mathbb{Q}[t].$$

Then $P_C(t) = (1-t)(1-qt) \left(\sum_{i=0}^{g+g^\perp-2} c_i t^i \right) + t^g$ and (21) can be written in the form

$$\begin{aligned} & \frac{\mathcal{W}_C(x, y) - x^n}{q-1} = \\ & \sum_{i=0}^{g+g^\perp-2} c_i \text{Coeff}_{t^{n-d-i}}([xt + y(1-t)]^n) + \text{Coeff}_{t^{n-d-g}} \left(\frac{[xt + y(1-t)]^n}{(1-t)(1-qt)} \right) = \\ & \sum_{i=0}^{g+g^\perp-2} c_i \left[\sum_{s=0}^n \binom{n}{s} x^{n-s} y^s \text{Coeff}_{t^{s-d-i}}(1-t)^s \right] + \frac{\mathcal{M}_{n,n-k+1}(x, y) - x^n}{q-1}, \end{aligned}$$

making use of (10). Note that $\text{Coeff}_{t^{s-d-i}}(1-t)^s = 0$ for $\forall s < d+i$, because $(1-t)^s$ has no pole at $t=0$ and $\binom{n}{s} \binom{s}{s-d-i} = \binom{n}{d+i} \binom{n-d-i}{s-d-i}$. As a result,

$$\begin{aligned} & \mathcal{W}_C(x, y) - \mathcal{M}_{n,n+1-k}(x, y) = \\ & \sum_{i=0}^{g+g^\perp-2} \sum_{s=d+i}^n (q-1) \binom{n}{s} \binom{s}{s-d-i} (-1)^{s-d-i} c_i x^{n-s} y^s = \\ & \sum_{i=0}^{g+g^\perp-2} \left[\sum_{s=d+i}^n \binom{n-d-i}{s-d-i} (-1)^{s-d-i} x^{n-s} y^{s-d-i} \right] (q-1) \binom{n}{d+i} c_i y^{d+i} = \\ & \sum_{i=0}^{g+g^\perp-2} \left[\sum_{j=0}^{n-d-i} \binom{n-d-i}{j} (-1)^j x^{n-d-i-j} y^j \right] \left[(q-1) \binom{n}{d+i} c_i y^{d+i} \right] = \\ & \sum_{i=0}^{g+g^\perp-2} \left[(q-1) \binom{n}{d+i} c_i \right] (x-y)^{n-d-i} y^{d+i}. \end{aligned}$$

Similar considerations for

$$\widetilde{D_{C^\perp}}(t) := \frac{P_{C^\perp}(t) - t^{g^\perp}}{(1-t)(1-qt)} = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i$$

provide (24). This shows the existence of polynomials $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i \in \mathbb{Q}[t]$, subject to (23), (24).

Towards the uniqueness of the polynomials $D_C(t)$, D_{C^\perp} , whose coefficients satisfy (23), (24), let us introduce $z := x - y$ and note that (23) is equivalent to

$$\sum_{i=0}^{g+g^\perp-2} \left[\binom{n}{d+i} (q-1) c_i \right] y^{d+i} z^{n-d-i} = \mathcal{W}_C(y+z, y) - \mathcal{M}_{n, n+1-k}(y+z, y) \quad (29)$$

and (24) amounts to

$$\sum_{i=0}^{g+g^\perp-2} \left[\binom{n}{d^\perp+i} (q-1) c_i^\perp \right] y^{d^\perp+i} z^{n-d^\perp-i} = \mathcal{W}_{C^\perp}(y+z, y) - \mathcal{M}_{n, k+1}(y+z, y). \quad (30)$$

Therefore (23), respectively, (24) determine uniquely $c_i \in \mathbb{Q}$, respectively, $c_i^\perp \in \mathbb{Q}$ for $\forall 0 \leq i \leq g+g^\perp-2$ and Duursma's reduced polynomials $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$, respectively, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i$ are related to the ζ -polynomials $P_C(t) = \sum_{i=0}^{g+g^\perp} a_i t^i$, respectively, $P_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp} a_i^\perp t^i$ by the equalities (25).

According to Proposition-Definition 7, Mac Williams identities for C , C^\perp are equivalent to the functional equation (13) for $P_C(t)$, $P_{C^\perp}(t)$. Making use of (25), one derives

$$\begin{aligned} D_C\left(\frac{1}{qt}\right) q^{g-1} t^{g+g^\perp-2} &= \left[\frac{P_C\left(\frac{1}{qt}\right) - \frac{1}{q^g t^g}}{\left(1 - \frac{1}{qt}\right) \left(1 - \frac{1}{t}\right)} \right] q^{g-1} t^{g+g^\perp-2} = \\ &= \left[\frac{P_C\left(\frac{1}{qt}\right) - \frac{1}{q^g t^g}}{(qt-1)(t-1)} \right] q^g t^{g+g^\perp} = \frac{P_C\left(\frac{1}{qt}\right) q^g t^{g+g^\perp} - t^{g^\perp}}{(1-t)(1-qt)} = \\ &= \frac{P_{C^\perp}(t) - t^{g^\perp}}{(1-t)(1-qt)} = D_{C^\perp}(t), \end{aligned}$$

which reveals that (13) is equivalent to (26).

Towards (27), let us denote $\rho_i := \binom{n}{d+i} (q-1) c_i$ for $0 \leq i \leq g+g^\perp-2$ and express

(29) in the form

$$\begin{aligned} \sum_{i=0}^{n-d-d^\perp} \rho_i y^{d+i} z^{n-d-i} &= \sum_{s=0}^n \left(\mathcal{W}_C^{(s)} - \mathcal{M}_{n,n+1-k}^{(s)} \right) (y+z)^{n-s} y^s = \\ &= \sum_{s=0}^n \left(\mathcal{W}_C^{(s)} - \mathcal{M}_{n,n+1-k}^{(s)} \right) \left(\sum_{i=0}^{n-s} \binom{n-s}{i} y^{n-i} z^i \right). \end{aligned}$$

After changing the summation index of the left hand side to $j := n-d-i$ and exchanging the summation order of the right hand side, one obtains

$$\sum_{j=d^\perp}^{n-d} \rho_{n-d-j} y^{n-j} z^j = \sum_{i=0}^n \left[\sum_{s=0}^{n-i} \binom{n-s}{i} \left(\mathcal{W}_C^{(s)} - \mathcal{M}_{n,n+1-k}^{(s)} \right) \right] y^{n-i} z^i.$$

The comparison of the coefficients of $y^{n-j} z^j$ in the above equality for $\forall d^\perp \leq j \leq n-d$ provides

$$\rho_{n-d-j} = \sum_{s=0}^{n-j} \binom{n-s}{j} \left(\mathcal{W}_C^{(s)} - \mathcal{M}_{n,n+1-k}^{(s)} \right) \quad \text{for } \forall d^\perp \leq j \leq n-d.$$

Changing the index to $i = n-d-j$, one derives (27). Similar considerations on (30) yields (28). □

In the case of \mathbb{F}_q -linear codes, (27) and (28) can be specified as follows:

Corollary 10. *Let $C \subset \mathbb{F}_q^n$ be an \mathbb{F}_q -linear code of minimum distance $d \geq 2$ with dual $C^\perp \subset \mathbb{F}_q^n$ of minimum distance $d^\perp \geq 2$ and*

$$D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i, \quad D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i \in \mathbb{Q}[t]$$

be Duursma's reduced polynomials of C, C^\perp . Then

$$\binom{n}{d+i} c_i \in \mathbb{Z} \quad \text{and} \quad \binom{n}{d^\perp+i} c_i^\perp \in \mathbb{Z} \tag{31}$$

are integers for all $0 \leq i \leq g+g^\perp-2$.

Proof. Note that the \mathbb{F}_q -linear structure of C, C^\perp induces actions

$$\mathbb{F}_q^* \times C \longrightarrow C, \quad (\lambda, a) \mapsto (\lambda a_1, \dots, \lambda a_n),$$

respectively,

$$\mathbb{F}_q^* \times C^\perp \longrightarrow C^\perp, \quad (\lambda, b) \mapsto (\lambda b_1, \dots, \lambda b_n),$$

preserving the Hamming weights. The orbit spaces $\mathbb{P}(C) := C \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^*$, respectively, $\mathbb{P}(C^\perp) := C^\perp \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^*$ are the corresponding projectivizations, which inherit the Hamming weight

$$\text{wt} : \mathbb{P}(\mathbb{F}_q^n) = \mathbb{P}^{n-1}(\mathbb{F}_q) \longrightarrow \{1, \dots, n\}, \quad \text{wt}([a]) := |\{1 \leq i \leq n \mid a_i \neq 0\}|$$

from the projectivization $\mathbb{P}(\mathbb{F}_q^n) = \mathbb{P}^{n-1}(\mathbb{F}_q)$ of the ambient vector space $\mathbb{F}_q^n \supset C, C^\perp$. The subsets $\mathbb{P}(C)^{(s)} := \{[a] \in \mathbb{P}(C) \mid \text{wt}([a]) = s\}$, $\mathbb{P}(C^\perp)^{(s)} := \{[b] \in \mathbb{P}(C^\perp) \mid \text{wt}([b]) = s\}$ are of cardinality

$$\left| \mathbb{P}(C)^{(s)} \right| = \frac{\mathcal{W}_C^{(s)}}{q-1} \in \mathbb{Z}^{\geq 0}, \quad \text{respectively,} \quad \left| \mathbb{P}(C^\perp)^{(s)} \right| = \frac{\mathcal{W}_{C^\perp}^{(s)}}{q-1} \in \mathbb{Z}^{\geq 0} \quad \text{for} \quad \forall 1 \leq s \leq n.$$

Baring in mind that the integers

$$\mathcal{M}_{n,n+1-k}^{(s)} = \binom{n}{s} (q-1) \left[\sum_{i=0}^{s-n-1+k} (-1)^i \binom{s-1}{i} q^{s-d-i} \right],$$

respectively,

$$\mathcal{M}_{n,k+1}^{(s)} = \binom{n}{s} (q-1) \left[\sum_{i=0}^{s-k-1} (-1)^i \binom{s-1}{i} q^{s-d-i} \right]$$

from (7) are divisible by $q-1$, one makes use of (27), (28) in order to derive (31). \square

4 Mac Williams identities for additive codes are equivalent to Polarized Riemann-Roch Conditions

In [13] we have shown that a formal power series $\zeta(t) = \sum_{m=0}^{\infty} \mathcal{A}_m t^m \in \mathbb{Z}[[t]]$ is a quotient $\zeta(t) = \frac{P(t)}{(1-t)(1-qt)}$ of a polynomial $P(t) \in \mathbb{Z}[t]$ exactly when $\zeta(t)$ is subject to the generic Riemann-Roch Conditions

$$\mathcal{A}_m = -q^{m+1} \text{Res}_{\frac{1}{q}}(\zeta(t)) - \text{Res}_1(\zeta(t)) \quad \text{for} \quad \forall m \geq \deg P(t) - 1$$

and the residuums $\text{Res}_{\frac{1}{q}}(\zeta(t))$, $\text{Res}_1(\zeta(t))$ of $\zeta(t)$ at its simple poles $\frac{1}{q}$, 1. By its very definition, the ζ -function $\zeta_C(t) = \frac{P_C(t)}{(1-t)(1-qt)} = \sum_{m=0}^{\infty} \mathcal{A}_m(C) t^m$ of an additive code $(C, +) < (G^n, +)$ is a quotient of a polynomial $P_C(t) \in \mathbb{Q}[t]$, so that satisfies the Generic Riemann-Roch Conditions

$$\mathcal{A}_m(C) = -q^{m+1} \text{ReS}_{\frac{1}{|G|}}(\zeta_C(t)) - \text{Res}_1(\zeta_C(t)) = \frac{q^{m+1} P_C\left(\frac{1}{|G|}\right) - P_C(1)}{|G| - 1}$$

for $\forall m \geq g + g^\perp - 1$, where g is the genus of C and g^\perp is the genus of C^\perp . The present section defines a more refined, polarized form of Riemann-Roch conditions and establishes

the equivalence of the Mac Williams identities for C , C^\perp to the polarized Riemann-Roch conditions on their ζ -functions.

The following lemma motivates the notion of Riemann-Roch Conditions for a formal power series of one variable.

Lemma 11. *Let $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ be a smooth irreducible curve of genus g , defined over a finite field \mathbb{F}_q and $\zeta_X(t) = \sum_{m=0}^{\infty} \mathcal{A}_m(X)t^m$ be the local Weil ζ -function of X . Then the Riemann-Roch Theorem on X implies the Riemann-Roch Conditions*

$$\mathcal{A}_m(X) = q^{m-g+1} \mathcal{A}_{2g-2-m}(X) + (q^{m-g+1} - 1) \text{Res}_1(\zeta_X(t)) \quad \text{for } \forall m \geq g,$$

where $\mathcal{A}_m(X)$ is the number of the effective divisors of degree m of the function field $\mathbb{F}_q(X)$ of X over \mathbb{F}_q and $\text{Res}_1(\zeta_X(t))$ is the residuum of $\zeta_X(t)$ at $t = 1$.

Proof. For an arbitrary divisor G of the function field $\mathbb{F}_q(X)$, let $H^0(X, \mathcal{O}_X([G]))$ be the space of the global sections of the line bundle, associated with G and $l(G) := \dim_{\mathbb{F}_q} H^0(X, \mathcal{O}_X([G]))$. Riemann-Roch Theorem asserts the existence of a canonical divisor K_X of degree $\deg K_X = 2g - 2$ with

$$l(G) = l(K_X - G) + \deg G - g + 1 \quad (32)$$

for all divisors G of $\mathbb{F}_q(X)$. In particular, if $\deg G > 2g - 2$ then

$$l(G) = \deg G - g + 1.$$

For any $k \in \mathbb{N}$ let $X(\mathbb{F}_{q^k}) := X \cap \mathbb{P}^N(\mathbb{F}_{q^k})$ be the set of the \mathbb{F}_{q^k} -rational points on X . The formal power series

$$\zeta_X(t) := \exp \left(\sum_{k=1}^{\infty} |X(\mathbb{F}_{q^k})| \frac{t^k}{k} \right) \in \mathbb{C}[[t]]$$

is called the local Weil ζ -function of X . It is well known (cf. Theorem 4.1.11 from [16]) that there is a ζ -polynomial $P_X(t) \in \mathbb{Z}[t]$ of $\deg P_X(t) = 2g$, such that

$$\zeta_X(t) = \frac{P_X(t)}{(1-t)(1-qt)}$$

and the residuum

$$\text{Res}_1(\zeta_X(t)) = \frac{P_X(1)}{q-1} = \frac{h}{q-1}$$

for the class number h of $\mathbb{F}_q(X)$. If G_1, \dots, G_h is a complete set of representatives of the linear equivalence classes of the divisors of $\mathbb{F}_q(X)$ of degree $m \in \mathbb{Z}^{\geq 0}$ then the divisors $K_X - G_1, \dots, K_X - G_h$ form a complete set of representatives of the linear equivalence classes of $\mathbb{F}_q(X)$ of degree $\deg(K_X - G_i) = 2g - 2 - m$. The effective divisors of $\mathbb{F}_q(X)$, which are linearly equivalent to G_i constitute the projective space $\mathbb{P}(H^0(X, \mathcal{O}_X([G_i]))) = \mathbb{P}^{l(G_i)-1}(\mathbb{F}_q)$. Thus, the number of the effective divisors of $\mathbb{F}_q(X)$ of degree m is

$$\mathcal{A}_m(X) = \sum_{i=1}^h \left| \mathbb{P}^{l(G_i)-1}(\mathbb{F}_q) \right| = \sum_{i=1}^h \frac{q^{l(G_i)} - 1}{q - 1}. \quad (33)$$

Substituting (32) in (33), one obtains

$$\mathcal{A}_m(X) = q^{m-g+1} \sum_{i=1}^h \left(\frac{q^{l(K_X - G_i)} - 1}{q - 1} \right) + h \left(\frac{q^{m-g+1} - 1}{q - 1} \right).$$

Bearing in mind that

$$\sum_{i=1}^h \left(\frac{q^{l(K_X - G_i)} - 1}{q - 1} \right) = \mathcal{A}_{2g-2-m}(X),$$

one concludes that

$$\mathcal{A}_m(X) = q^{m-g+1} \mathcal{A}_{2g-2-m}(X) + h \left(\frac{q^{m-g+1} - 1}{q - 1} \right) \quad \text{for } \forall m \geq 0. \quad (34)$$

Note that in the case of $g \geq 2$ the relations (34) with $0 \leq m \leq g - 2$ are equivalent to are equivalent to the ones with index $g \leq 2g - 2 - m \leq 2g - 2$ and (34) is trivial for $m = g - 1$. If $g = 0$ then $X = \mathbb{P}^1(\overline{\mathbb{F}}_q)$ is the projective line and the equalities (34) reduce to

$$\mathcal{A}_m(X) = \frac{q^{m+1} - 1}{q - 1} \quad \text{for } \forall m \geq 0.$$

When $g = 1$, the curve X is elliptic and

$$\mathcal{A}_m(X) = h \left(\frac{q^m - 1}{q - 1} \right)$$

for the class number h and all $m \geq 1$. □

Definition 12. A formal power series $\zeta(t) = \sum_{m=0}^{\infty} \mathcal{A}_m t^m \in \mathbb{C}[[t]]$ satisfies the Riemann-Roch Conditions $\text{RRC}_q(g)$ with base $q \in \mathbb{N}$ of genus $g \in \mathbb{Z}^{\geq 0}$ if

$$\mathcal{A}_m = q^{m-g+1} \mathcal{A}_{2g-2-m} + (q^{m-g+1} - 1) \text{Res}_1(\zeta(t)) \quad \text{for } \forall m \geq g$$

and the residuum $\text{Res}_1(\zeta(t))$ of $\zeta(t)$ at $t = 1$.

Here is a polarized version of the Riemann-Roch Conditions.

Definition 13. Formal power series $\zeta(t) = \sum_{m=0}^{\infty} \mathcal{A}_m t^m \in \mathbb{C}[[t]]$ and $\zeta^\perp(t) = \sum_{m=0}^{\infty} \mathcal{A}_m^\perp t^m \in \mathbb{C}[[t]]$ satisfy the Polarized Riemann-Roch Conditions $\text{PRRC}_q(g, g^\perp)$ of genera $g, g^\perp \in \mathbb{Z}^{\geq 0}$ with base $q \in \mathbb{N}$ if

$$\mathcal{A}_m = q^{m-g+1} \mathcal{A}_{g+g^\perp-2-m}^\perp + (q^{m-g+1} - 1) \text{Res}_1(\zeta(t)) \quad \text{for } \forall m \geq g,$$

$$\mathcal{A}_{g-1} = \mathcal{A}_{g^\perp-1}^\perp \quad \text{and}$$

$$\mathcal{A}_m^\perp = q^{m-g^\perp+1} \mathcal{A}_{g+g^\perp-2-m} + (q^{m-g^\perp+1} - 1) \text{Res}_1(\zeta^\perp(t)) \quad \text{for } \forall m \geq g^\perp,$$

where $\text{Res}_1(\zeta(t))$, $\text{Res}_1(\zeta^\perp(t))$ stand for the corresponding residuums at $t = 1$.

One can view the Riemann-Roch Theorem on a smooth irreducible projective curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\mathbb{F}_q)$ as a quantitative expression of the Serre duality on X . Thus, $\text{RRC}_q(g)$ and, therefore, $\text{PRRC}_q(g, g^\perp)$ may be interpreted as a Serre duality between the formal power series $\zeta(t), \zeta^\perp(t)$.

Observe also that $\text{PRRC}_q(g, g^\perp)$ implies

$$\mathcal{A}_m = \kappa_1 q^m + \kappa_2, \quad \mathcal{A}_m^\perp = \kappa_1^\perp q^m + \kappa_2^\perp \quad \text{for } \forall m \geq g + g^\perp - 1$$

and some constants $\kappa_j, \kappa_j^\perp \in \mathbb{C}$. These are equivalent to the recurrence relations

$$\mathcal{A}_{m+2} - (q+1)\mathcal{A}_{m+1} + q\mathcal{A}_m = \mathcal{A}_{m+2}^\perp - (q+1)\mathcal{A}_{m+1}^\perp + q\mathcal{A}_m^\perp = 0 \quad \text{for } \forall m \geq g + g^\perp - 1$$

and hold exactly when

$$\zeta(t) = \frac{P(t)}{(1-t)(1-qt)} \quad \text{and} \quad \zeta^\perp(t) = \frac{P^\perp(t)}{(1-t)(1-qt)}$$

for polynomials $P(t), P^\perp(t)$. Thus, the Polarized Riemann-Roch Conditions with base q require $\zeta(t), \zeta^\perp(t)$ to be rational functions of t with denominators $(1-t)(1-qt)$ and imply the Generic Riemann-Roch Conditions on $\zeta(t)$ and on $\zeta^\perp(t)$. Note that the Generic Riemann-Roch Conditions for $\zeta(t) = \frac{P(t)}{(1-t)(1-qt)}$ coincide with the Riemann-Roch Conditions $\text{RRC}_q(g)$ with $m \geq 2g - 1$ if and only if

$$P\left(\frac{1}{q}\right) = \left(\frac{1}{q}\right)^g P(1).$$

While preparing the present article, we came up with Randriambololona's article [18] on Harder-Narasimhan theory, Serre duality and Riemann-Roch Theorem for linear codes. Our main Theorem 14 reveals that the Riemann-Roch Theorem 44 from [18] is stronger than our Polarized Riemann-Roch Conditions $\text{PRRC}_q(g, g^\perp)$ from Definition 13. More precisely, for an arbitrary \mathbb{F}_q -linear $[n, k, d]$ -code $C \subset \mathbb{F}_q^n$ of genus $g = n + 1 - d - k$ and an arbitrary subset $J \subseteq [n] := \{1, \dots, n\}$, Randriambololona defines the cohomology group $H^0(C, J) := C \cap \mathbb{F}_q^J$ as the largest linear subspace of C with support J . If $C^\perp \subset \mathbb{F}_q^n$ is the dual code of C and $[n] \setminus J$ is the complement of J , then the Riemann-Roch Theorem 44 from [18] asserts that

$$\dim_{\mathbb{F}_q} H^0(C, J) - \dim_{\mathbb{F}_q} H^0(C^\perp, [n] \setminus J) = (|J| - d) - g + 1. \quad (35)$$

The author mentions that (35) implies the functional equation

$$\zeta_{C^\perp}(t) = \zeta_C\left(\frac{1}{qt}\right) q^{g-1} t^{g+g^\perp-2} \quad (36)$$

of Duursma's ζ -functions $\zeta_C(t), \zeta_{C^\perp}(t)$ of C, C^\perp . As far as (36) is equivalent to Mac Williams identities for the weight distribution of C, C^\perp , our Theorem 14 reveals that Randriambololona's Riemann-Roch Theorem 44 from [18] implies our Polarized Riemann-Roch Conditions $\text{PRRC}_q(g, g^\perp)$ on $\zeta_C(t), \zeta_{C^\perp}(t)$.

Here is the main result of the present article, which interprets Mac Williams duality on additive codes as Polarized Riemann-Roch Conditions or as a polarized form of the Serre duality on a smooth irreducible projective curve, defined over a finite field.

Theorem 14. *Mac Williams identities for the weight distribution of an additive code $(C, +) < (G^n, +)$ of minimum distance $d \geq 2$ and genus g with dual $(C^\perp, \cdot) < (\widehat{G}^n, \cdot)$ of minimum distance $d^\perp \geq 2$ and genus g^\perp are equivalent to the Polarized Riemann-Roch Conditions $\text{PRRC}_{|G|}(g, g^\perp)$ on their ζ -functions $\zeta_C(t)$, $\zeta_{C^\perp}(t)$.*

Proof. Let us denote by $q := |G|$ the order of G . First, we prove the theorem for $g, g^\perp \in \mathbb{N}$. If $\zeta_C(t) := \frac{P_C(t)}{(1-t)(1-qt)} = \sum_{m=0}^{\infty} \mathcal{A}_m(C)t^m$ for some $\mathcal{A}_m(C) \in \mathbb{Q}$ then

$$D_C(t) = \zeta_C(t) - \frac{t^g}{(1-t)(1-qt)} = \sum_{m=0}^{\infty} \mathcal{A}_m(C)t^m - \sum_{m=g}^{\infty} \left(\frac{q^{m-g+1} - 1}{q - 1} \right) t^m \quad (37)$$

by

$$\begin{aligned} \frac{t^g}{(1-t)(1-qt)} &= t^g \left(\sum_{i=0}^{\infty} t^i \right) \left(\sum_{j=0}^{\infty} q^j t^j \right) = \\ &= \sum_{m=g}^{\infty} (q^{m-g} + q^{m-g-1} + \dots + q + 1) t^m = \sum_{m=g}^{\infty} \left(\frac{q^{m-g+1} - 1}{q - 1} \right) t^m \in \mathbb{Z}[[t]]. \end{aligned}$$

Thus, $D_C(t) - \zeta_C(t)$ is a sum of terms of degree $\geq g$,

$$\begin{aligned} D_C(t) &= \sum_{m=0}^{g-2} \mathcal{A}_m(C)t^m + \mathcal{A}_{g-1}(C)t^{g-1} + \sum_{i=g}^{g+g^\perp-2} c_i t^i = \\ &= \sum_{m=0}^{g-2} \mathcal{A}_m(C)t^m + \mathcal{A}_{g-1}(C)t^{g-1} + \left(\sum_{m=0}^{g^\perp-2} c_{g+g^\perp-2-m} t^{-m} \right) t^{g+g^\perp-2} \end{aligned} \quad (38)$$

and, respectively,

$$D_{C^\perp}(t) = \sum_{m=0}^{g^\perp-2} \mathcal{A}_m(C^\perp)t^m + \mathcal{A}_{g^\perp-1}(C^\perp)t^{g^\perp-1} + \left(\sum_{m=0}^{g-2} c_{g+g^\perp-2-m}^\perp t^{-m} \right) t^{g+g^\perp-2}$$

for $\zeta_{C^\perp}(t) := \frac{P_{C^\perp}(t)}{(1-t)(1-qt)} = \sum_{m=0}^{\infty} \mathcal{A}_m(C^\perp)t^m$. According to

$$\begin{aligned} D_C \left(\frac{1}{qt} \right) q^{g-1} t^{g+g^\perp-2} &= \\ &= \sum_{m=0}^{g-2} \mathcal{A}_m(C) q^{g-1-m} t^{g+g^\perp-2-m} + \mathcal{A}_{g-1}(C) t^{g^\perp-1} + \sum_{m=0}^{g^\perp-2} c_{g+g^\perp-2-m} q^{m-g^\perp+1} t^m = \\ &= \sum_{m=0}^{g^\perp-2} c_{g+g^\perp-2-m} q^{m-g^\perp+1} t^m + \mathcal{A}_{g-1}(C) t^{g^\perp-1} + \left(\sum_{m=0}^{g-2} q^{-m+g-1} \mathcal{A}_m(C) t^{-m} \right) t^{g+g^\perp-2}, \end{aligned}$$

Mac Williams identities (26) for Duursma's reduced polynomials of a pair $C, C^\perp \subset \mathbb{F}_q^n$ of mutually dual linear codes of genus $g \geq 1$, respectively, $g^\perp \geq 1$ amount to

$$c_{g+g^\perp-2-m} = q^{-m+g^\perp-1} \mathcal{A}_m(C^\perp) \quad \text{for } \forall 0 \leq m \leq g^\perp - 2, \quad (39)$$

$$\mathcal{A}_{g^\perp-1}(C^\perp) = \mathcal{A}_{g-1}(C) \quad \text{and} \quad (40)$$

$$c_{g+g^\perp-2-m}^\perp = q^{-m+g-1} \mathcal{A}_m(C) \quad \text{for } \forall 0 \leq m \leq g - 2. \quad (41)$$

Substituting $m = g + g^\perp - 2 - i$ and making use of (37), one observes that (39) is equivalent to

$$\mathcal{A}_i(C) = q^{i-g+1} \mathcal{A}_{g+g^\perp-2-i}(C^\perp) + \left(\frac{q^{i-g+1} - 1}{q - 1} \right) \quad \text{for } \forall g \leq i \leq g + g^\perp - 2.$$

Exchanging C with C^\perp , one expresses (41) in the form

$$\mathcal{A}_i(C^\perp) = q^{i-g^\perp+1} \mathcal{A}_{g+g^\perp-2-i}(C) + \left(\frac{q^{i-g^\perp+1} - 1}{q - 1} \right) \quad \text{for } \forall g^\perp \leq i \leq g + g^\perp - 2.$$

According to (37),

$$\mathcal{A}_i(C) = \frac{q^{i-g+1} - 1}{q - 1} \quad \text{for } \forall i \geq g + g^\perp - 1.$$

Similarly,

$$\mathcal{A}_i(C^\perp) = \frac{q^{i-g^\perp+1} - 1}{q - 1} \quad \text{for } \forall i \geq g + g^\perp - 1.$$

Bearing in mind that $\zeta_C(t)$ and $\zeta_{C^\perp}(t)$ have no pole at $t = 0$, one introduces $\mathcal{A}_{-j}(C) = \mathcal{A}_{-j}(C^\perp) = 0$ for $\forall j \in \mathbb{N}$ and expresses Mac Williams identities in the form

$$\mathcal{A}_i(C) = q^{i-g+1} \mathcal{A}_{g+g^\perp-2-i}(C^\perp) + \left(\frac{q^{i-g+1} - 1}{q - 1} \right) \quad \text{for } \forall i \geq g, \quad (42)$$

$$\mathcal{A}_{g^\perp-1}(C^\perp) = \mathcal{A}_{g-1}(C) \quad \text{and} \quad (43)$$

$$\mathcal{A}_i(C^\perp) = q^{i-g^\perp+1} \mathcal{A}_{g+g^\perp-2-i}(C) + \left(\frac{q^{i-g^\perp+1} - 1}{q - 1} \right) \quad \text{for } \forall i \geq g^\perp. \quad (44)$$

Note also that the rational function

$$\zeta_C(t) = \frac{P_C(t)}{(1-t)(1-qt)}$$

has residuum

$$\text{Res}_1(\zeta_C(t)) = \frac{P_C(1)}{q-1} = \frac{1}{q-1}$$

at 1. Thus, for $g \geq 1$ and $g^\perp \geq 1$, Mac Williams identities (42), (43), (44) for $C, C^\perp \subset \mathbb{F}_q^n$ are equivalent to the polarized Riemann-Roch conditions $\text{PRRC}(g, g^\perp)$.

In the case of $g = 0$, one has $|C| = |G|^{n+1-d} < |G|^n$ by $d \geq 2$. Thus, Lemma 3 applies to provide $g^\perp = 0$. The ζ -functions

$$\zeta_C(t) = \zeta_{C^\perp}(t) = \frac{1}{(1-t)(1-qt)} = \zeta_{\mathbb{P}^1(\overline{\mathbb{F}}_q)}(t)$$

coincide with the ζ -function of the projective line $\mathbb{P}^1(\overline{\mathbb{F}}_q)$ and satisfy the Riemann-Roch Conditions $\text{RRC}_{|G|}(0)$ of genus 0, which are equivalent to the Polarized Riemann-Roch Conditions $\text{PRRC}_{|G|}(0, 0)$.

The ζ -functions

$$\zeta_C(t) = \zeta_{C^\perp}(t) = \frac{1}{(1-t)(1-qt)} = \zeta_{\mathbb{P}^1(\overline{\mathbb{F}}_q)}(t)$$

coincide with the ζ -function of the projective line $\mathbb{P}^1(\overline{\mathbb{F}}_q)$ and satisfy the Riemann-Roch Conditions $\text{RRC}(0)$ of genus $g = 0$, which are equivalent to the Polarized Riemann-Roch Conditions $\text{PRRC}(0, 0)$.

□

As a byproduct of the proof of Theorem 14 we obtain the following

Corollary 15. *The lower parts $\varphi_C(t) = \sum_{i=0}^{g-2} c_i t^i$, $\varphi_{C^\perp}(t) = \sum_{i=0}^{g^\perp-2} c_i^\perp t^i$ of Duursma's reduced polynomials $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i$ of an additive code $(C, +) < (G^n, +)$ of minimum distance $d \geq 2$ and genus $g \geq 1$ and its dual $(C^\perp, \cdot) < (\widehat{G}^n, \cdot)$ of minimum distance $d^\perp \geq 2$ and genus $g^\perp \geq 1$, together with the number $c_{g-1} = c_{g^\perp-1}^\perp \in \mathbb{Q}$ determine uniquely*

$$D_C(t) = \varphi_C(t) + c_{g-1} t^{g-1} + \varphi_{C^\perp} \left(\frac{1}{qt} \right) q^{g^\perp-1} t^{g+g^\perp-2}, \quad (45)$$

$$D_{C^\perp}(t) = \varphi_{C^\perp}(t) + c_{g-1} t^{g^\perp-1} + \varphi_C \left(\frac{1}{qt} \right) q^{g-1} t^{g+g^\perp-2}. \quad (46)$$

Proof. The substitution of (39) in (38) yields

$$D_C(t) = \varphi_C(t) + c_{g-1} t^{g-1} + \left(\sum_{m=0}^{g^\perp-2} c_m^\perp q^{-m} t^{-m} \right) q^{g^\perp-1} t^{g+g^\perp-2},$$

whereas (45). Replacing C by C^\perp , C^\perp by C and $c_{g^\perp-1}^\perp$ by c_{g-1} , one obtains (46).

□

5 Averaging, algebraic-geometric and probabilistic interpretations of the coefficients of Duursma's reduced polynomial

Let G be a finite abelian group, $(C, +) \leq (G^n, +)$ be an additive code. Abbreviate $[n] := \{1, \dots, n\}$ and denote by $\binom{[n]}{s}$ the collection of the subsets $\alpha = \{\alpha_1, \dots, \alpha_s\} \subseteq [n]$ of cardinality $1 \leq s \leq n$. We proceed by an averaging interpretation of the lower parts of Duursma's reduced polynomials of C and C^\perp .

Proposition 16. *Let $(C, +) < (G^n, +)$ be an additive code of minimum distance $d \geq 2$ and genus $g \geq 1$ with dual $(C^\perp, \cdot) < (\widehat{G}^n, \cdot)$ of minimum distance $d^\perp \geq 2$ and genus $g^\perp \geq 1$. Denote by $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i \in \mathbb{Q}[t]$ Duursma's reduced polynomials of these codes and put*

$$(C \setminus \{0_G^n\})^{(\subseteq \gamma)} := \{a \in C \setminus \{0_G^n\} \mid \text{Supp}(a) \subseteq \gamma\},$$

respectively,

$$(C^\perp \setminus \{\varepsilon^n\})^{(\subseteq \gamma)} := \{\pi \in C^\perp \setminus \{\varepsilon^n\} \mid \text{Supp}(\pi) \subseteq \gamma\}$$

for $\gamma \in \binom{[n]}{s}$. Then

$$(|G| - 1)c_i = \binom{n}{d+i}^{-1} \left(\sum_{\gamma \in \binom{[n]}{d+i}} |(C \setminus \{0_G^n\})^{(\subseteq \gamma)}| \right) \quad \text{for } \forall 0 \leq i \leq g-1 \quad (47)$$

is the average cardinality of an intersection of $C \setminus \{0_G^n\}$ with $n-d-i$ coordinate hyperplanes in $(G^n, +)$ and

$$(|G| - 1)c_i^\perp = \binom{n}{d^\perp+i}^{-1} \left(\sum_{\gamma \in \binom{[n]}{d^\perp+i}} |(C^\perp \setminus \{\varepsilon^n\})^{(\subseteq \gamma)}| \right) \quad \text{for } \forall 0 \leq i \leq g^\perp-1 \quad (48)$$

is the average cardinality of an intersection of C^\perp with $n-d^\perp-1$ coordinate hyperplanes in (\widehat{G}^n, \cdot) .

Proof. The equality (47) will be derived by counting the disjoint union

$$U^{(d+i)} := \coprod_{\gamma \in \binom{[n]}{d+i}} (C \setminus \{0_G^n\})^{(\subseteq \gamma)}$$

in two different ways. Namely, a word $a \in C \setminus \{0_G^n\}$ of weight $\text{wt}(a) = s \in \mathbb{N}$ has support $\sigma := \text{Supp}(a) \subseteq \gamma$ for some $\gamma \in \binom{[n]}{d+i}$ if and only if the complements $\neg\gamma := [n] \setminus \gamma \subseteq [n] \setminus \sigma =: \neg\sigma \in \binom{[n]}{n-s}$ are subject to the opposite inclusion. There are exactly $\binom{n-s}{n-d-i}$ such $\neg\gamma$ for $|\neg\gamma| = n-d-i \leq n-s$ and none of such $\neg\gamma$ for $n-d-i > n-s$. Thus,

any $a \in C \setminus \{0_G^n\}$ of $\text{wt}(a) = s \leq d + i$ is counted $\binom{n-s}{n-d-i}$ times in $U^{(d+i)}$ and noone $a \in C \setminus \{0_G^n\}$ with $\text{wt}(a) = s > d + i$ is counted in $U^{(d+i)}$. That justifies the equality

$$|U^{(d+i)}| = \sum_{s=1}^{d+i} \binom{n-s}{n-d-i} \mathcal{W}_C^{(s)} \quad \text{for } \forall 0 \leq i \leq n-d.$$

According to (27) from Proposition 9,

$$(|G| - 1) \binom{n}{d+i} c_i = \sum_{s=1}^{d+i} \mathcal{W}_C^{(s)} \binom{n-s}{n-d-i} \quad \text{for } \forall d \leq d+i \leq n-k,$$

due to the vanishing of $\mathcal{M}_{n,n+1-k}^{(s)} = 0$ for $\forall 1 \leq s \leq d+i \leq n-k$. As a result,

$$|U^{(d+i)}| = (|G| - 1) \binom{n}{d+i} c_i \quad \text{for } 0 \leq i \leq n-k-d = g-1.$$

Combining with

$$|U^{(d+i)}| = \sum_{\gamma \in \binom{[n]}{d+i}} |(C \setminus \{0_G^n\})^{(\subseteq \gamma)}| \quad \text{for } 0 \leq i \leq n-d,$$

one justifies (47) for $\forall 0 \leq i \leq g-1$. Similar considerations on $\coprod_{\gamma \in \binom{[n]}{d+i}} (C^\perp \setminus \{0_G^n\})^{(\subseteq \gamma)}$

provide (48) for $\forall 0 \leq i \leq g^\perp - 1$.

□

In the case of \mathbb{F}_q -linear codes C , $C^\perp \subset \mathbb{F}_q^n$, when the finite sets $(C \setminus \{0_{\mathbb{F}_q}^n\})^{(\subseteq \gamma)}$, $(C^\perp \setminus \{0_{\mathbb{F}_q}^n\})^{(\subseteq \gamma)}$ are invariant under componentwise multiplications by $\lambda \in \mathbb{F}_q^*$, the following corollary formulates the result in terms of the cardinalities of the subsets

$$\mathbb{P}(C)^{(\subseteq \gamma)} := \{[a] \in \mathbb{P}(C) \mid \text{Supp}([a]) \subseteq \gamma\},$$

respectively,

$$\mathbb{P}(C^\perp)^{(\subseteq \gamma)} := \{[a] \in \mathbb{P}(C^\perp) \mid \text{Supp}([a]) \subseteq \gamma\}$$

of the projectivizations $\mathbb{P}(C) := C \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^* \subset \mathbb{P}(\mathbb{F}_q^n) := \mathbb{F}_q^n \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^*$, respectively, $\mathbb{P}(C^\perp) := C^\perp \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^* \subset \mathbb{P}(\mathbb{F}_q^n)$, viewed as projective subspaces of the projectivization $\mathbb{P}(\mathbb{F}_q^n)$ of \mathbb{F}_q^n .

Pellikaan, Shen and van Wee have shown in [17] that an arbitrary \mathbb{F}_q -linear code $C \subset \mathbb{F}_q^n$ has an algebraic-geometric representation. It means an existence of a smooth irreducible projective curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$, defined over \mathbb{F}_q , distinct \mathbb{F}_q -rational points $P_1, \dots, P_n \in X(\mathbb{F}_q) := X \cap \mathbb{P}^N(\mathbb{F}_q)$ and a divisor G of the function field $\mathbb{F}_q(X)$, such that $\text{Supp}(G) \cap \text{Supp}(D) = \emptyset$ and $C = \mathcal{E}_D \mathcal{L}_X(G)$ is the image of the evaluation map

$$\mathcal{E}_D : \mathcal{L}_X(g) = H^0(X, \mathcal{O}_X([G])) \longrightarrow \mathbb{F}_q^n, \quad \mathcal{E}_D(f) = (f(P_1), \dots, f(P_n))$$

at $D := P_1 + \dots + P_n$. Any algebraic-geometric realization $C = \mathcal{E}_D \mathcal{L}_X(G)$ of C is associated with a algebraic-geometric realization $C^\perp = \mathcal{E}_D \mathcal{L}_X(K_X - G + d)$ of the dual codes $C^\perp \subset \mathbb{F}_q^n$, where K_X stands for a canonical divisor of $\mathbb{F}_q(X)$. From now on, we denote by $l(E) := \dim_{\mathbb{F}_q} \mathcal{L}_X(E)$ the dimension of $\mathcal{L}_X(E)$. The next proposition interprets the elements of the projectivizations

$$\mathbb{P}(C) := C \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^* \quad \text{rm and} \quad \mathbb{P}(C)^{(\subseteq \gamma)} := \mathbb{P}\left(\left(C \setminus \{0_{\mathbb{F}_q}^n\}\right)\right)$$

with $\gamma \in \binom{[n]}{s}$, $d \leq s \leq n - k$ as orbits of effective divisors of $\mathbb{F}_q(X)$.

Corollary 17. *Let $C = \mathcal{E}_D \mathcal{L}_X(G)$ be an algebraic-geometric representation of an \mathbb{F}_q -linear code $C \subset \mathbb{F}_q^n$ of minimum distance $d \geq 2$ and genus $g \geq 1$ with dual $C^\perp = \mathcal{E}_D \mathcal{L}_X(K_X - G + d)$ of minimum distance $d^\perp \geq 2$ and genus $g^\perp \geq 1$. Denote by*

$$\text{Div}_{\geq 0}(\sim G, D) := \{E = G + \text{div}(f) \geq 0 \mid f \in \mathbb{F}_q(X), \text{Supp}(D) \not\subseteq \text{Supp}(E)\}$$

the set of the effective divisors of $\mathbb{F}_q(X)$, which are linearly equivalent to G and do not contain $\{P_1, \dots, P_n\}$ in its support and put $D_\delta := \sum_{i \in \delta} P_i$ for $\forall \delta \in \binom{[n]}{s}$, $1 \leq s \leq n$.

(i) Then the kernel $\ker \mathcal{E}_D = \mathcal{L}_X(G - D)$ of the surjective \mathbb{F}_q -linear evaluation map $\mathcal{E}_D : \mathcal{L}_X(G) \rightarrow C$ acts on $\text{Div}_{\geq 0}(\sim G, D)$ by the rule

$$\mathcal{L}_X(G - D) \times \text{Div}_{\geq 0}(\sim G, D) \longrightarrow \text{Div}_{\geq 0}(\sim G, D), \quad (g, G = \text{div}(f)) \mapsto G + \text{div}(f + g) \quad (49)$$

and the projectivization

$$\mathbb{P}(C) = \text{Div}_{\geq 0}(\sim G, D) / \mathcal{L}_X(G - D)$$

of C is the orbit space of $\text{Div}_{\geq 0}(\sim G, D)$ under this action.

In particular, if $m < n$ then $\mathbb{P}(C) = \text{Div}_{\geq 0}(\sim G, D)$.

(ii) If $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$ is Duursma's reduced polynomial of $C \subset \mathbb{F}_q^n$ and $\zeta_C(t) = \sum_{i=0}^{\infty} \mathcal{A}_i(C) t^i$ is the ζ -function of C then

$$c_i = \mathcal{A}_i(C) = \binom{n}{d+i}^{-1} \left(\sum_{\delta \in \binom{[n]}{n-d-i}} |\text{Div}_{\geq 0}(\sim (G - D_\delta), D) / \mathcal{L}_X(G - D)| \right) =$$

$$\binom{n}{d+i}^{-1} q^{-l(G-D)} \left(\sum_{\delta \in \binom{[n]}{n-d-i}} |\text{Div}_{\geq 0}(\sim (G - D_\delta), D)| \right) \quad \text{with} \quad 0 \leq i \leq g-1$$

is the average cardinality of an $\mathcal{L}_X(G - D)$ -orbit space of

$$\text{Div}_{\geq 0}(\sim (G - D_\delta), D) := \{E = G - D_\delta + \text{div}(f) \geq 0 \mid f \in \mathbb{F}_q(X), \text{Supp}(D) \not\subseteq \text{Supp}(E)\}$$

with $\delta \in \binom{[n]}{n-d-i}$.

(iii) If $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i$ is Duursma's reduced polynomial of C^\perp and $\zeta_{C^\perp}(t) = \sum_{i=0}^{\infty} \mathcal{A}_i(C^\perp) t^i$ is the ζ -function of C^\perp then

$$c_i^\perp = \mathcal{A}_i(C^\perp) = \left(\binom{n}{d^\perp + i} \right)^{-1} \left(\sum_{\gamma \in \binom{[n]}{d^\perp + i}} |\text{Div}_{\geq 0}(\sim (K_X - G + D_\gamma), D) / \mathcal{L}_X(K_X - G)| \right) =$$

$$\left(\binom{n}{d^\perp + i} \right)^{-1} q^{-l(K_X - G)} \left(\sum_{\gamma \in \binom{[n]}{d^\perp + i}} |\text{Div}_{\geq 0}(\sim (K_X - G + D_\gamma), D)| \right)$$

with $0 \leq i \leq g^\perp - 1$, is the average cardinality of an $\mathcal{L}_X(K_X - G)$ -orbit space of

$$\text{Div}_{\geq 0}(\sim (K_X - G + D_\gamma), D) := \{E = K_X - G + D_\gamma + \text{div}(f) \geq 0 \mid f \in \mathbb{F}_q(X), \text{Supp}(D) \not\subseteq \text{Supp}(E)\}$$

with $\gamma \in \binom{[n]}{d^\perp + i}$.

(iv) The coefficients of the ζ -function $\zeta_C(t) = \sum_{i=0}^{\infty} \mathcal{A}_i(C) t^i$ of C have

$$\mathcal{A}_i(C) \binom{n}{d+i} \in \mathbb{Z}^{\geq 0} \quad \text{for } 0 \leq i \leq g + g^\perp - 2 \quad \text{and}$$

$$\mathcal{A}_i(C) \in \mathbb{Z}^{\geq 0} \quad \text{for } \forall i > g + g^\perp - 2.$$

Proof. (i) First of all, one has to check that the kernel of $\mathcal{E}_D : \mathcal{L}_X(G) \rightarrow \mathcal{E}_D \mathcal{L}_X(G) = C$ equals $\mathcal{L}_X(G - D)$. If $G = G_+ - G_-$ for effective divisors G_+, G_- of $\mathbb{F}_q(X)$ then $f \in \mathcal{L}_X(G)$ exactly when $\text{div}(f)_0 + G_+ \geq \text{div}(f)_\infty + G_-$. Due to the disjointness of the supports of $\text{div}(f)_0, \text{div}(f)_\infty$ and of G_+, G_- , this is equivalent to the conditions $\text{div}(f)_\infty \leq G_+$ and $\text{div}(f)_0 \geq G_-$. Now, $f \in \mathcal{L}_X(G)$ belongs to $\ker \mathcal{E}_D$ if and only if $D \leq \text{div}(f)_0$. By assumption, $\text{Supp}(G_-) \cap \text{Supp}(D) \subseteq \text{Supp}(G) \cap \text{Supp}(D) = \emptyset$, so that the kernel of \mathcal{E}_D on $\mathcal{L}_X(G)$ consists of the rational functions $f \in \mathbb{F}_q(X)$ with $\text{div}(f)_\infty \leq G_+$ and $\text{div}(f)_0 \geq G_- - D$. That, in turn, amounts to $G - D + \text{div}(f) = G_+ - G_- - D + \text{div}(f)_0 - \text{div}(f)_\infty \geq 0$ and reveals that $\ker \mathcal{E}_D = \mathcal{L}_X(G - D)$. Now, $\mathcal{E}_D : \mathcal{L}_X(G) \rightarrow C = \mathcal{E}_D \mathcal{L}_X(G)$ restricts to a surjective map of sets

$$\mathcal{E}_D : \mathcal{L}_X(G) \setminus \ker \mathcal{E}_D = \mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D) \longrightarrow C \setminus \{0_{\mathbb{F}_q}^n\}.$$

The correspondence

$$G + \text{div} : \mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D) \longrightarrow \text{Div}_{\geq 0}(\sim G, D), \quad f \mapsto G + \text{div}(f),$$

associating to $f \in \mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D)$ the effective divisor $G + \text{div}$, linearly equivalent to G , which does not contain $\text{Supp}(D) = \{P_1, \dots, P_n\}$ in its support, coincides with the quotient map of $\mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D)$ with respect to the \mathbb{F}_q^* -action

$$\mathbb{F}_q^* \times (\mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D)) \longrightarrow (\mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D)), \quad (\lambda, f) \mapsto \lambda f.$$

Note that \mathbb{F}_q^* acts on $C \setminus \{0_{\mathbb{F}_q}^n\}$ by the rule

$$\mathbb{F}_q^* \times (C \setminus \{0_{\mathbb{F}_q}^n\}) \longrightarrow C \setminus \{0_{\mathbb{F}_q}^n\}, \quad (\lambda, (c_1, \dots, c_n)) \mapsto (\lambda c_1, \dots, \lambda c_n)$$

and denote by

$$\eta_{\mathbb{F}_q^*} : C \setminus \{0_{\mathbb{F}_q}^n\} \longrightarrow \mathbb{P}(C) := C \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^*$$

the projectivization map of C . Straightforwardly verification establishes the \mathbb{F}_q^* -equivalence of $\mathcal{E}_D : \mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D) \longrightarrow C \setminus \{0_{\mathbb{F}_q}^n\}$, i.e., $\lambda \mathcal{E}_D(f) = \lambda(f(P_1), \dots, f(P_n)) = (\lambda f(P_1), \dots, \lambda f(P_n))$, $\forall f \in \mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D)$. Therefore \mathcal{E}_D induces a surjective map of finite sets

$$\overline{\mathcal{E}_D} : \text{Div}_{\geq 0}(\sim G, D) \longrightarrow \mathbb{P}(C),$$

closing the commutative diagram

$$\begin{array}{ccc} \mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D) & \xrightarrow{\mathcal{E}_D} & C \setminus \{0_{\mathbb{F}_q}^n\} \\ \downarrow G + \text{div} & & \downarrow \eta_{\mathbb{F}_q^*} \\ \text{Div}_{\geq 0}(\sim G, D) & \xrightarrow{\overline{\mathcal{E}_D}} & \mathbb{P}(C) \end{array} .$$

The corresponding fibres of $G + \text{div}$ and $\eta_{\mathbb{F}_q^*}$ are isomorphic to each other and to \mathbb{F}_q^* , so that the fibres of $\overline{\mathcal{E}_D} : \text{Div}_{\geq 0}(\sim G, D) \rightarrow \mathbb{P}(C)$ are isomorphic to $\ker \mathcal{E}_D = \mathcal{L}_X(G - D)$. For arbitrary $g \in \mathcal{L}_X(G - D)$ and $G + \text{div}(f) \in \text{Div}_{\geq 0}(\sim G, D)$, note that $G + \text{div}(f + g) \in \text{Div}_{\geq 0}(\sim G, D)$, as far as the assumption $\text{Supp}(D) \subseteq \text{Supp}(G + \text{div}(f + g))$ implies $\text{Supp}(D) \subseteq \text{div}(f + g)_0$, due to $\text{div}(f + g)_\infty \leq G_+$ and $\text{Supp}(D) \cap \text{Supp}(G) = \emptyset$. Then $0_{\mathbb{F}_q}^n = \mathcal{E}_D(f + g) = \mathcal{E}_D(f) + \mathcal{E}_D(g) = \mathcal{E}_D(f)$, which contradicts $f \notin \mathcal{L}_X(G - D) = \ker \mathcal{E}_D$ and verifies the correctness of the map (49). All fibres of (49) are isomorphic to the linear system $\mathcal{L}_X(G - D)$, because the assumption $\text{div}(f + g) = \text{div}(f + h)$ for some $g, h \in \mathcal{L}_X(G - D)$ requires $f + h = \lambda(f + g)$ for some $\lambda \in \mathbb{F}_q^*$ and amounts to $(\lambda - 1)f = h - \lambda g \in \mathcal{L}_X(G - D) = \ker \mathcal{E}_D$. The choice of $f \in \mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D)$ specifies that $\lambda = 1$ and $g = h$. Since all the fibres of $\overline{\mathcal{E}_D}$ are isomorphic to $\mathcal{L}_X(G - D)$,

$$\mathbb{P}(C) = \overline{\mathcal{E}_D} \text{Div}_{\geq 0}(\sim G, D) = \text{Div}_{\geq 0}(\sim G, D) / \mathcal{L}_X(G - D)$$

can be viewed as the quotient space of $\text{Div}_{\geq 0}(\sim G, D)$, under the action of $\mathcal{L}_X(G - D)$.

In particular, if $m < n$ then $\deg(G - D) < 0$ and $\mathcal{L}_X(G - D) = \{0_{\mathbb{F}_q(X)}\}$. That allows to identify $\mathbb{P}(C) = \text{Div}_{\geq 0}(\sim G, D)$ with the effective divisors, linearly equivalent to G , which do not contain $\{P_1, \dots, P_n\}$ in its support.

(ii) Note that the support function $\text{Supp} : C \rightarrow \{0, 1, \dots, n\}$ is invariant under the action

$$\mathbb{F}_q^* \times C \longrightarrow C, \quad (\lambda, (c_1, \dots, c_n)) \mapsto (\lambda c_1, \dots, \lambda c_n)$$

of \mathbb{F}_q^* and descends to $\text{Supp } \mathbb{P}(C) \rightarrow \{1, \dots, n\}$ with $\text{Supp}([a]) = \text{Supp} \eta_{\mathbb{F}_q^*}(a) = \text{Supp}(a)$ for $\forall a \in C \setminus \{0_{\mathbb{F}_q}^n\}$. In particular,

$$\eta_{\mathbb{F}_q^*} : \left(C \setminus \{0_{\mathbb{F}_q}^n\} \right)^{(\subseteq \gamma)} \longrightarrow \mathbb{P}(C)^{(\subseteq \gamma)} := \{[a] \in \mathbb{P}(C) \mid \text{Supp}([a]) \subseteq \gamma\}$$

is an unramified \mathbb{F}_q^* -Galois covering and (47) takes the form

$$c_i = \binom{n}{d+i}^{-1} \left(\sum_{\gamma \in \binom{[n]}{d+i}} |\mathbb{P}(C)^{(\subseteq \gamma)}| \right) \quad \text{for } \forall 0 \leq i \leq g-1,$$

according to $\left| \left(C \setminus \{0_{\mathbb{F}_q}^n\} \right)^{(\subseteq \gamma)} \right| = (q-1) |\mathbb{P}(C)^{(\subseteq \gamma)}|$. If $\delta := \neg \gamma = [n] \setminus \gamma \in \binom{[n]}{n-d-i}$ is the complement of $\gamma \in \binom{[n]}{d+i}$, it suffices to show that

$$\mathbb{P}(C)^{(\subseteq \gamma)} \simeq \text{Div}_{\geq 0}(\sim (G - D_\delta), D) / \mathcal{L}_X(G - D) \quad (50)$$

is the orbit space of

$$\text{Div}_{\geq 0}(\sim (G - D_\delta), D) := \{E = G - D_\delta + \text{div}(f) \geq 0 \mid f \in \mathbb{F}_q(X), \text{Supp}(D) \not\subseteq \text{Supp}(E)\}$$

under the action

$$\mathcal{L}_X(G - D) \times \text{Div}_{\geq 0}(\sim (G - D_\delta), D) \longrightarrow \text{Div}_{\geq 0}(\sim (G - D_\delta), D),$$

$$(g, G - D_\delta + \text{div}(f)) \mapsto G - D_\delta + \text{div}(f + g)$$

of $\mathcal{L}_X(G - D)$, in order to conclude the proof of (ii). To this end, note that

$$\begin{aligned} \mathcal{E}_D^{-1} \left(C^{(\subseteq \gamma)} \setminus \{0_{\mathbb{F}_q}^n\} \right) &= \{f \in \mathcal{L}_X(G) \setminus \mathcal{L}_X(G - D) \mid \text{Supp}(D_\delta) \leq \text{div}(f)_0\} = \\ &= \mathcal{L}_X(G - D_\delta) \setminus \mathcal{L}_X(G - D). \end{aligned}$$

The considerations from (i), applied to $G - D_\delta$ instead of G provide the commutative diagram

$$\begin{array}{ccc} \mathcal{L}_X(G - D_\delta) \setminus \mathcal{L}_X(G - D) & \xrightarrow{\mathcal{E}_D} & \left(C \setminus \{0_{\mathbb{F}_q}^n\} \right)^{(\subseteq \gamma)} \\ \downarrow G - D_\delta + \text{div} & & \downarrow \eta_{\mathbb{F}_q^*} \\ \text{Div}_{\geq 0}(\sim (G - D_\delta), D) & \xrightarrow{\overline{\mathcal{E}_D}} & \mathbb{P}(C)^{(\subseteq \gamma)} \end{array},$$

where $\overline{\mathcal{E}_D}$ is a surjective map, whose fibres are isomorphic to $\mathcal{L}_X(G - D)$. That allows the identification (??) with $|\text{Div}_{\geq 0}(\sim (G - D_\delta), D)| = q^{l(G-D)} |\mathbb{P}(C)^{(\subseteq \gamma)}|$.

(iii) follows from (48) by observing that

$$\begin{aligned}\mathbb{P}(C^\perp)^{(\subseteq \gamma)} &= \mathbb{P}(\mathcal{E}_D \mathcal{L}_X(K_X - G + D))^{(\subseteq \gamma)} \simeq \\ \text{Div}_{\geq 0}(\sim K_X - G + D - (D - D_\gamma), D) / \mathcal{L}_X(G - D) &= \\ \text{Div}_{\geq 0}(\sim (K_X - G + D_\gamma), D) / \mathcal{L}_X(G - D) &\text{ for } \forall \gamma \in \binom{[n]}{d^\perp + i}.\end{aligned}$$

(iv) Note that (ii) implies $\mathcal{A}_i(C) \binom{n}{d+i} \in \mathbb{Z}^{\geq 0}$ for $\forall 0 \leq i \leq g-1$ and (iii) guarantees that $\mathcal{A}_i(C^\perp) \binom{n}{d^\perp+i} \in \mathbb{Z}^{\geq 0}$ for $\forall 0 \leq i \leq g^\perp-1$. Making use of (42), one concludes that

$$\mathcal{A}_i(C) \binom{n}{d+i} = q^{i-g+1} \mathcal{A}_{g+g^\perp-2-i}(C^\perp) \binom{n}{d+i} + \binom{n}{d+i} \left(\frac{q^{i-g+1}-1}{q-1} \right)^\epsilon \mathbb{Z}^{\geq 0}$$

for $\forall g \leq i \leq g+g^\perp-2$, according to $d^\perp + (g+g^\perp-2-i) = n-d-i$ and $\binom{n}{n-d-i} = \binom{n}{d+i}$. In the case of $i > g+g^\perp-2$, (42) reduces to

$$\mathcal{A}_i(C) = \frac{q^{i-g+1}-1}{q-1} \in \mathbb{Z}^{\geq 0}.$$

□

By Theorem 1.1.28 and Exercise 1.1.29 from [19], the homogeneous weight enumerator of an \mathbb{F}_q -linear code $C \subset \mathbb{F}_q^n$ with dual C^\perp of minimum distance $d^\perp \geq 2$ can be expressed in the form

$$\mathcal{W}_C(x, y) = x^n + \sum_{i=0}^{n-d} B_i (x-y)^i y^{n-i} \quad \text{with} \quad B_i = (q-1) \left(\sum_{\alpha \in \binom{[n]}{i}} |\mathbb{P}(C)^{(\subseteq -\alpha)}| \right).$$

Corollary ?? reveals that Tsfasman-Vlăduț-Nogin's coefficients $B_{d+i} = \binom{n}{d+i} (q-1) c_i$ are closely related with the coefficients c_i of Duursma's reduced polynomial $D_C(t)$ of C for $\forall 0 \leq i \leq g-1$.

Proposition 18. *Let $(C, +) < (G^n, +)$ be an additive code of minimum distance $d \geq 2$ and genus $g \geq 1$ with dual $(C^\perp, \cdot) < (\widehat{G}^n, \cdot)$ of minimum distance $d^\perp \geq 2$ and genus $g^\perp \geq 1$. Suppose that $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i \in \mathbb{Q}[t]$ are Duursma's reduced polynomials of C , C^\perp , $p_C^{(s)}$ (respectively, $\pi_{C^\perp}^{(s)}$) are the probabilities of $a \in G^n$ (respectively, $\pi \in \widehat{G}^n$) of weight $\text{wt}(a) = s$ (respectively, $\text{wt}(\pi) = s$) to belong to C (respectively, to C^\perp) and $\overline{p_a^{(s)}}$ (respectively, $\overline{p_\pi^{(s)}}$) are the probabilities of $\gamma \in \binom{[n]}{s}$ to contain the support $\text{Supp}(a)$ of $a \in C$ (respectively, the support $\text{Supp}(\pi)$ of $\pi \in C^\perp$). Then*

$$(i) \quad c_i = \sum_{s=d}^{d+i} p_C^{(s)} \binom{d+i}{s} (|G|-1)^{s-1} \quad \text{for } \forall 0 \leq i \leq g-1, \quad (51)$$

$$c_i = |G|^{i-g+1} \left[\sum_{s=d^\perp}^{n-d-i} p_{C^\perp}^{(s)} \binom{n-d-i}{s} (|G|-1)^{s-1} \right] \quad \text{for } \forall g \leq i \leq g+g^\perp-2; \quad (52)$$

$$(ii) \quad c_i = (|G| - 1)^{-1} \left(\sum_{a \in C \setminus \{0_G^n\}} \overline{p_a^{(d+i)}} \right) \quad \text{for } \forall 0 \leq i \leq g - 1, \quad (53)$$

$$c_i = (|G| - 1)^{-1} |G|^{i-g+1} \left(\sum_{\pi \in C^\perp \setminus \{\varepsilon^n\}} \overline{p_\pi^{(n-d-i)}} \right) \quad \text{for } \forall g \leq i \leq g + g^\perp - 2. \quad (54)$$

Proof. (i) If $(G^n)^{(s)} := \{a \in G^n \mid \text{wt}(a) = s\}$, respectively, $C^{(s)} := \{a \in C \mid \text{wt}(a) = s\}$ are the subsets of the words of weight $1 \leq s \leq n$, $q := |G|$ and $\mathcal{W}_C^{(s)} := |C^{(s)}|$ then the probability of $a \in (G^n)^{(s)}$ to belong to C is

$$p_C^{(s)} = \frac{|C^{(s)}|}{|(G^n)^{(s)}|} = \frac{\mathcal{W}_C^{(s)}}{\binom{[n]}{s} (q-1)^s}.$$

According to (27) for Proposition 9, if $0 \leq i \leq g - 1 = n - d - k$ then

$$(q-1) \binom{n}{d+i} c_i = \sum_{s=d}^{d+i} \mathcal{W}_C^{(s)} \binom{n-s}{n-d-i} \quad (55)$$

as far as $\mathcal{W}_C^{(0)} = \mathcal{M}_{n,n+1-k}^{(0)} = 1$, $\mathcal{W}_C^{(s)} = 0$ for $\forall 1 \leq s \leq d-1$ and $\mathcal{M}_{n,n+1-k}^{(s)} = 0$ for $\forall 1 \leq s \leq d+i \leq n-k$. Substituting $\mathcal{W}_C^{(s)} = \binom{n}{s} (q-1)^s p_C^{(s)}$ in (55) and making use of $\binom{n}{d+i}^{-1} \binom{n}{s} \binom{n-s}{n-d-i} = \binom{d+i}{s}$, one concludes that (51).

The application of (51) to C^\perp yields

$$c_i^\perp = \sum_{s=d^\perp}^{d^\perp+i} p_{C^\perp}^{(s)} \binom{d^\perp+i}{s} (q-1)^{s-1} \quad \text{for } \forall 0 \leq i \leq g^\perp - 1.$$

According to (45) from Corollary (15), Duursma's reduced polynomial $D_C(t)$ can be represented as

$$D_C(t) = \sum_{i=0}^{g-1} c_i t^i + \sum_{i=g}^{g+g^\perp-2} c_{g+g^\perp-2-i}^\perp q^{i-g+1} t^i.$$

Therefore

$$c_i = q^{i-g+1} c_{g+g^\perp-2-i}^\perp \quad \text{for } \forall g \leq i \leq g + g^\perp - 2. \quad (56)$$

Plugging in (53) with $0 \leq g + g^\perp - 2 - i = n - d - d^\perp - i \leq g^\perp - 2$ in the above formula, one obtains (52).

(ii) If $a \in C$ has support $\text{Supp}(a) \in \binom{[n]}{s}$ and $s \leq w \leq n$, then the number of $\gamma \in \binom{[n]}{w}$, containing $\text{Supp}(a)$ equals $\binom{n-s}{w-s}$. Thus, the probability of $\gamma \in \binom{[n]}{w}$ to contain $\text{Supp}(a)$ equals $\overline{p_a^{(w)}} = \frac{\binom{n-s}{w-s}}{\binom{n}{w}}$. If $\text{wt}(a) = s > w$ then $\overline{p_a^{(w)}} = 0$. Making use of this, one represents (55) as

$$c_i = \sum_{s=1}^{d+i} \frac{\mathcal{W}_C^{(s)} \binom{n-s}{d+i-s}}{(q-1) \binom{n}{d+i}} = \sum_{s=1}^{d+i} \sum_{a \in C^{(s)}} \frac{\overline{p_a^{(d+i)}}}{q-1} = (q-1)^{-1} \left(\sum_{a \in C \setminus \{0_G^n\}} \overline{p_b^{(d^\perp+i)}} \right)$$

for $\forall 0 \leq i \leq g-1$. Combining with (56), one derives (54). \square

In the case of \mathbb{F}_q -linear codes, Proposition 18 specializes to the following

Corollary 19. *Let $C \subset \mathbb{F}_Q^n$ be an \mathbb{F}_q -linear code of minimum distance $d \geq 2$ and genus $g \geq 1$ with dual $C^\perp \subset \mathbb{F}_q^n$ of minimum distance $d^\perp \geq 2$ and genus $g^\perp \geq 1$. Denote by $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i \in \mathbb{Q}[t]$ Duursma's reduced polynomials of C , C^\perp , put $\pi_{\mathbb{P}(C)}^{(s)}$ (respectively, $\pi_{\mathbb{P}(C^\perp)}^{(s)}$) for the probability of $[a] \in \mathbb{P}^{n-1}(\mathbb{F}_q)$ to belong to $\mathbb{P}(C)$ (respectively, to $\mathbb{P}(C^\perp)$) and designate by $\overline{\pi_{[a]}^{(s)}}$ (respectively, by $\overline{\pi_{[b]}^{(s)}}$) the probability of $\gamma \in \binom{[n]}{s}$ to contain the support of $[a] \in \mathbb{P}(C)$ (respectively, of $[b] \in \mathbb{P}(C^\perp)$). Then:*

$$(i) \quad c_i = \sum_{s=d}^{d+i} \pi_{\mathbb{P}(C)}^{(s)} \binom{d+i}{s} (q-1)^{s-1} \quad \text{for } \forall 0 \leq i \leq g-1,$$

$$c_i = q^{i-g+1} \left[\sum_{s=d^\perp}^{n-d-i} \pi_{\mathbb{P}(C^\perp)}^{(s)} \binom{n-d-i}{s} (q-1)^{s-1} \right] \quad \text{for } \forall g \leq i \leq g+g^\perp-2;$$

$$(ii) \quad c_i = \sum_{[a] \in \mathbb{P}(C)} \overline{\pi_{[a]}^{(d+i)}} \quad \text{for } \forall 0 \leq i \leq g-1,$$

$$c_i = q^{i-g+1} \left(\sum_{[b] \in \mathbb{P}(C^\perp)} \overline{\pi_{[b]}^{(n-d-i)}} \right) \quad \text{for } \forall g \leq i \leq g+g^\perp-2.$$

Proof. (i) It suffices to note that $\mathbb{P}(C)^{(s)} := \{[a] \in \mathbb{P}(C) \mid \text{wt}([a]) = s\}$ is of cardinality

$$|\mathbb{P}(C)^{(s)}| = \frac{|C^{(s)}|}{|\mathbb{F}_q^*|} = \frac{\mathcal{W}_C^{(s)}}{q-1} \quad \text{for } \forall 1 \leq s \leq n$$

and $\mathbb{P}^{n-1}(\mathbb{F}_q)^{(s)} := \{[a] \in \mathbb{P}^{n-1}(\mathbb{F}_q) \mid \text{wt}(a) = s\}$ is of cardinality

$$|\mathbb{P}^{n-1}(\mathbb{F}_q)^{(s)}| = \binom{n}{s} \frac{|(\mathbb{F}_q^*)^s|}{|\mathbb{F}_q^*|} = \binom{n}{s} (q-1)^{s-1},$$

so that

$$\pi_{\mathbb{P}(C)}^{(s)} = \frac{|\mathbb{P}(C)^{(s)}|}{|\mathbb{P}^{n-1}(\mathbb{F}_q)^{(s)}|} = \frac{\mathcal{W}_C^{(s)}}{\binom{n}{s} (q-1)^s} = p_C^{(s)}$$

and (i) is an immediate consequence of (51) and (52) from Proposition 18.

(ii) The first equality follows from (53) by noting that the support of $a \in C \setminus \{0_{\mathbb{F}_q}^n\}$ is constant along an \mathbb{F}_q^* -orbit on $C \setminus \{0_{\mathbb{F}_q}^n\}$ and the projectivization $\mathbb{P}(C) := C \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^*$ of C is the \mathbb{F}_q^* -orbit space of $C \setminus \{0_{\mathbb{F}_q}^n\}$. The second equality follows from (54), $\mathbb{P}(C^\perp) := C^\perp \setminus \{0_{\mathbb{F}_q}^n\} / \mathbb{F}_q^*$ and the fact that the weight is constant along the \mathbb{F}_q^* -orbits on $C^\perp \setminus \{0_{\mathbb{F}_q}^n\}$. \square

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